FINITE LOCALITIES II

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Introduction

This is Part II of the series that began with [Ch1] - and the reader is assumed to be familiar with Part I. References to results in Part I will be made by prefixing a "I" to the number of the cited result. Thus, for example, lemma I.3.12 is Stellmacher's splitting lemma.

Let $(\mathcal{L}, \Delta, S)$ be a locality. There is then a category $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$ - the "fusion system" of $\mathcal{L}$ - whose objects are the subgroups of S, and whose morphisms are compositions of conjugation maps $c_g : X \to Y$ from one subgroup of S into another, induced by elements $g \in \mathcal{L}$. We say that $\mathcal{L}$ is a locality *on* $\mathcal{F}$.

There is an extensive theory of abstract fusion systems and, more particularly, of "saturated" fusion systems. The references by Craven [Cr], and by Aschbacher, Kessar, and Oliver [AKO] provide far more material than will be needed here. In fact, we shall provide a self-contained treatment of fusion systems, up to a certain point. Thus, there will be introductory material in sections 1 and 2, but then in section 6 we shall make use of some powerful theorems from [5a], and exploit some arguments from [He2], in order to obtain the main results we shall need concerning fusion systems of localities. The material at the beginning of section 1, through definition 1.8, is all that is required in order to put in place the notion of "proper locality" (It should be mentioned that the definition of "radical" subgroup in 1.8 is different from the usual one.)

The locality $(\mathcal{L}, \Delta, S)$ on $\mathcal{F}$ is defined to be proper if, firstly, Δ is not too small - the technical condition being that Δ should contain the set $\mathcal{F}^{cr}$ of all subgroups of S which are both centric and radical in $\mathcal{F}$. Secondly (and lastly), what is required is that all of the normalizer groups $N_{\mathcal{L}}(P)$ for $P \in \Delta$ should be of characteristic p - where one says that a finite group G is of characteristic p if $C_G(O_p(G)) \leq O_p(G)$. It turns out (see Proposition 2.9) that if $\mathcal{L}$ is an arbitrary locality for which Δ is not too small in the above sense, and also not too large, then $\mathcal{L}$ has a canonical homomorphic image which is a proper locality on $\mathcal{F}$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

We shall be concerned here almost exclusively with proper localities, with the aim of providing the technical back-ground for the main theorems in Part III. Three questions need to be addressed, concerning the fusion system $\mathcal{F}$ of a proper locality $(\mathcal{L}, \Delta, S)$.

(1) Can one determine all of the proper localities on $\mathcal{F}$, up to isomorphism ?

(2) If $\mathcal{L}$ and $\mathcal{L}'$ are proper localities on $\mathcal{F}$, then what is the relationship between the set $\mathfrak{N}(\mathcal{L})$ of partial normal subgroups of $\mathcal{L}$ and the set $\mathfrak{N}(\mathcal{L}')$ of partial normal subgroups of $\mathcal{L}'$?

(3) What special properties does $\mathcal{F}$ possess, by virtue of its being the fusion system of a proper locality ?

The answers are given by Theorems A1 and A2, and by the results on fusion systems in section 6. In order to state these results we need the following terminology.

A non-empty collection Γ of subgroups of S is *$\mathcal{F}$-closed* if $Q \in \Gamma$ whenever there exists an $\mathcal{F}$-homomorphism $\phi : P \to Q$ for some $P \in \Gamma$. It will be shown in 6.– that the set $\mathcal{F}^s$ of $\mathcal{F}$-subcentric subgroups of S (defined in 1.8) is $\mathcal{F}$-closed.

Definition. Let $(\mathcal{L}, \Delta, S)$ be a locality on $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$. Then $\mathcal{L}$ is *proper* if:

(PL1) $P \in \Delta$ for every subgroup P of S such that P is both centric and radical in $\mathcal{F}$.

(PL2) The groups $N_{\mathcal{L}}(P)$ for $P \in \Delta$ are of characteristic p.

Let $(\mathcal{L}, \Delta, S)$ is a proper locality on $\mathcal{F}$. Lemma 2.8 will show that Δ is necessarily an $\mathcal{F}$-closed subset of $\mathcal{F}^s$. It is a straightforward exercise with the definitions (see lemma 2.11) to show that if Δ_0 is an $\mathcal{F}$-closed subset of Δ containing $\mathcal{F}^{cr}$, then there is a unique proper locality $(\mathcal{L}_0, \Delta_0, S)$ on $\mathcal{F}$ such that the partial group $\mathcal{L}_0$ is a subset of $\mathcal{L}$. We shall call $\mathcal{L}_0$ the *restriction* of $\mathcal{L}$. Theorems A1 and A2 concern the opposite sort of operation, by which one expands, rather than restricts, the set of objects.

Theorem A1. *Let $(\mathcal{L}, \Delta, S)$ be a proper locality on $\mathcal{F}$ and let Δ^+ be an $\mathcal{F}$-closed collection of subgroups of S such that $\Delta \subseteq \Delta^+ \subseteq \mathcal{F}^s$.*

(a) *There exists a proper locality $(\mathcal{L}^+, \Delta^+, S)$ on $\mathcal{F}$ such that $\mathcal{L}$ is the restriction $\mathcal{L}^+ |_\Delta$ of $\mathcal{L}^+$ to Δ. Moreover, $\mathcal{L}^+$ is generated by $\mathcal{L}$ as a partial group.*

(b) *For any proper locality $(\widetilde{\mathcal{L}}, \Delta^+, S)$ on $\mathcal{F}$ whose restriction to Δ is $\mathcal{L}$, there is a unique isomorphism $\mathcal{L}^+ \to \widetilde{\mathcal{L}}$ which restricts to the identity map on $\mathcal{L}$.*

Recall that a partial subgroup $\mathcal{N}$ of a partial group $\mathcal{L}$ is *normal* in $\mathcal{L}$ (or is a *partial normal subgroup* of $\mathcal{L}$, denoted $\mathcal{N} \trianglelefteq \mathcal{L}$) if $x^g := g^{-1}xg \in \mathcal{N}$ for all $x \in \mathcal{N}$ and $g \in \mathcal{L}$ for which the product $g^{-1}xg$ is defined. Recall also: for any partial group $\mathcal{L}$ and any subset X of $\mathcal{L}$, $\langle X \rangle$ is defined to be the intersection of the set of partial subgroups of $\mathcal{L}$ containing X. The intersection of partial subgroups is again a partial subgroup by I.1.8, and $\langle X \rangle$ is called the partial subgroup generated by X.

Theorem A2. *Let the hypothesis and notation be as in Theorem A1, let $\mathcal{N}$ be a partial normal subgroup of $\mathcal{L}$, and set $T = S \cap \mathcal{N}$. Let $X := \mathcal{N}^{\mathcal{L}^+}$ be the set of all elements of $\mathcal{L}^+$ of the form f^h, where $f \in \mathcal{N}$, $h \in \mathcal{L}^+$, and where the product $f^h = h^{-1}fh$ is defined in $\mathcal{L}^+$. Let $\mathcal{N}^+ = \langle X \rangle$ be the partial subgroup of $\mathcal{L}^+$ generated by X. Then:*

(*) *$\mathcal{N}^+ \trianglelefteq \mathcal{L}^+$ and $\mathcal{L} \cap \mathcal{N}^+ = \mathcal{N}$. In particular $S \cap \mathcal{N}^+ = T$.*

Further, the mapping $\mathcal{N} \mapsto \mathcal{N}^+$ is a bijection from the set of partial normal subgroups of $\mathcal{L}$ to the set of partial normal subgroups of $\mathcal{L}^+$, and the inverse mapping is given by $\mathcal{N}^+ \mapsto \mathcal{L} \cap \mathcal{N}^+$.

The notions of fully normalized subgroup V of $\mathcal{F}$, the fusion systems $N_{\mathcal{F}}(V)$ and $C_{\mathcal{F}}(V)$, and of a fusion system being (cr)-generated, are given in 1.4 and 1.10.

Theorem B. *Let $\mathcal{F}$ be the fusion system of a proper locality $(\mathcal{L}, \Delta, S)$, and let V be a subgroup of S such that V is fully normalized in $\mathcal{F}$. Then the following hold.*

(a) *For each $\mathcal{F}$-conjugate U of V, there exists an $\mathcal{F}$-homomorphism $\phi : N_S(U) \to S$ such that $U\phi = V$.*

(b) *Both $N_{\mathcal{F}}(V)$ and $C_{\mathcal{F}}(V)$ are (cr)-generated.*

The proof of Theorem B is given in section 6 (where it appears as Theorem 6.1), and relies on a full panoply of deep results concerning so-called "saturated" fusion systems. Indeed, the proof consists in first showing that $\mathcal{F}$ is saturated, and in then obtaining (a) and (b) as corollaries to the known results. This is not an entirely satisfactory approach, for two reasons. The first is that the notion of saturated fusion system will play no role whatsoever in this series, other than in its role in the proof of Theorem B. Since the conditions (a) and (b) in Theorem B turn out to be the key properties of $\mathcal{F}$, rather than the conditions defining saturation, we would have preferred to omit the notion of saturated fusion system altogether.

The second reason is that, as mentioned in the introduction to Part I, this series of papers concerns only the "finite case" of a much more general, parallel series being prepared with Alex Gonzalez, and for which the standard notion of "saturation" turns out to be inappropriate. A proof of a generalized version of Theorem B from first principles will be carried out in the parallel series, but it is not easy to justify burdening the reader with such a proof here.

Readers who are willing to accept Theorems A and B can, if they wish, ignore the proofs (sections 3 through 5, and the beginning of section 6, through 6.3) entirely. The remainder of section 6 concerns the set $\mathcal{F}^s$ of subcentric subgroups, where $\mathcal{F}$ is the fusion system of a proper locality. This material is essentially taken from Henke's work [He2] (where the focus is on saturated fusion systems). Since section 1 is largely a review of basic material on fusion systems, there will be readers who may wish to simply skim, or even skip, section 1. For those readers we should mention that there are a few notions pertaining to fusion systems in general which have been reformulated here. In particular, the definition of "centric radical" subgroup given here in 1.8 is not the standard one (though it is equivalent to the standard definition in the case of a saturated fusion system). With this small proviso concerning section 1, readers who are in a hurry to get to the meat of things in Part III are advised to read section 2 for the basic material on proper localities, and to then skip ahead to the final section 7. Section 7 concerns the notions of $O^p_{\mathcal{L}}(\mathcal{N})$ and $O^{p'}_{\mathcal{L}}(\mathcal{N})$ for $\mathcal{N}$ a partial normal subgroup of a proper locality $\mathcal{L}$.

Section 1: Fusion systems

We begin this section by providing a brief summary of some of the terminology, and some of the basic results, pertaining to general fusion systems. Some of the definitions are non-standard, but turn out to be equivalent to the standard definitions in the case of the fusion system of a proper locality.

Definition 1.1. Let S be a finite p-group. A *fusion system* $\mathcal{F}$ on S is a category, whose set of objects is the set of subgroups of S, and whose morphisms satisfy the following conditions (in which P and Q are subgroups of S).

(1) Each $\mathcal{F}$-morphism $P \to Q$ is an injective homomorphism of groups.
(2) If $g \in S$ and $P^g \leq Q$ then the conjugation map $c_g : P \to Q$ is an $\mathcal{F}$-morphism.
(3) If $\phi : P \to Q$ is an $\mathcal{F}$-morphism then the bijection $P \to Im(\phi)$ defined by ϕ is an $\mathcal{F}$-isomorphism.

One most often refers to $\mathcal{F}$-morphisms as $\mathcal{F}$-homomorphisms, in order to emphasize condition (1). Notice that (2) implies that all inclusion maps between subgroups of S are $\mathcal{F}$-homomorphisms, and hence the restriction of an $\mathcal{F}$-homomorphism $P \to Q$ to a subgroup of P is again an $\mathcal{F}$-homomorphism.

Let G be a finite group and let S be a p-subgroup of G. There is then a fusion system $\mathcal{F} = \mathcal{F}_S(G)$ on S in which the $\mathcal{F}$-homomorphisms $P \to Q$ are the maps $c_g : P \to Q$ given by conjugation by those elements $g \in G$ for which $P^g \leq Q$.

Definition 1.2. Let $\mathcal{F}$ be a fusion system on S, and let $\mathcal{F}'$ be a fusion system on S'. A homomorphism $\alpha : S \to S'$ is a *fusion-preseving* (relative to $\mathcal{F}$ and $\mathcal{F}'$) if, for each $\mathcal{F}$-homomorphism $\phi : P \to Q$, there exists an $\mathcal{F}'$-homomorphism $\psi : P\alpha \to Q\alpha$ such that $\alpha\mid_P \circ \psi = \phi \circ \alpha \mid_Q$.

Notice that each of the $\mathcal{F}'$-homomorphisms ψ in the preceding definition is uniquely determined, since all $\mathcal{F}'$-homomorphisms are injective. Thus, if $\alpha : S \to S'$ is a fusion-preserving homomorphism then α induces a mapping $Hom_{\mathcal{F}}(P, Q) \to Hom_{\mathcal{F}'}(P\alpha, Q\alpha)$, for each pair (P, Q) of subgroups of S. The following result is then easily verifed.

Lemma 1.3. *Let $\mathcal{F}$ be a fusion system on S, let $\mathcal{F}'$ be a fusion system on S', and let $\alpha : S \to S'$ be a fusion-preserving homomorphism. Then the mapping $P \mapsto P\alpha$ from objects of $\mathcal{F}$ to objects of $\mathcal{F}'$, together with the set of mappings*

$$\alpha_{P,Q} : Hom_{\mathcal{F}}(P, Q) \to Hom_{\mathcal{F}'}(P\alpha, Q\alpha) \quad (P, Q \leq S)$$

defines a functor $\alpha^* : \mathcal{F} \to \mathcal{F}'$. $\square$

In view of the preceding result, a fusion-preserving homomorphism α may also be called a *homomorphism of fusion systems*. Notice that the inverse of a fusion-preserving isomorphism is fusion-preserving, and is therefore an isomorphism of fusion systems.

In the special case where $S \leq S'$ and the inclusion map $S \to S'$ is fusion-preserving, we say that $\mathcal{F}$ is a *fusion subsystem* of $\mathcal{F}'$. Thus, $\mathcal{F}_S(S)$ is a fusion subsystem of $\mathcal{F}$ for each fusion system $\mathcal{F}$ on S, by 1.1(2). We refer to $\mathcal{F}_S(S)$ as the *trivial fusion system* on S.

If $\mathcal{F}$ is a fusion system on S and P is a subgroup of S, write $P^{\mathcal{F}}$ for the set of subgroups of S of the form $P\phi$, $\phi \in Hom_{\mathcal{F}}(P, S)$. The elements of $P^{\mathcal{F}}$ are the *$\mathcal{F}$-conjugates* of P.

For the remainder of this section let $\mathcal{F}$ be a fixed fusion system on the finite p-group S.

Definition 1.4. Let $P \leq S$ be a subgroup of S. Then P is *fully normalized* in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(P')|$ for all $P' \in P^{\mathcal{F}}$. Similarly, P is *fully centralized* in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for all $P' \in P^{\mathcal{F}}$.

Let $U \leq S$ be a subgroup of S. The *normalizer* $N_{\mathcal{F}}(U)$ of U in $\mathcal{F}$ is the category whose objects are the subgroups of $N_S(U)$, and whose morphisms $P \to Q$ (P and Q subgroups of $N_S(U)$) are restrictions of $\mathcal{F}$-homomorphisms $\phi : PU \to QU$ such that $U\phi = U$. Similarly, the *centralizer* $C_{\mathcal{F}}(U)$ of U in $\mathcal{F}$ is the category whose objects are the subgroups of $C_S(U)$ and whose morphisms $\phi : P \to Q$ are restrictions of $\mathcal{F}$-homomorphisms $\phi : PU \to QU$ such that ϕ induces the identity map on U. One observes that $N_{\mathcal{F}}(U)$ is a fusion system on $N_S(U)$ and that $C_{\mathcal{F}}(U)$ is a fusion system on $C_S(U)$. The following result is immediate from 1.3.

Lemma 1.5. *Let $U, V \leq S$ be subgroups of S, and suppose that there exists an $\mathcal{F}$-isomorphism $\alpha : N_S(U) \to N_S(V)$ such that $U\alpha = V$. Then α is an isomorphism $N_{\mathcal{F}}(U) \to N_{\mathcal{F}}(V)$. Similarly, if $\beta : C_S(U)U \to C_S(V)V$ is an $\mathcal{F}$-isomorphism such that $U\beta = V$, then the restriction of β to $C_S(U)$ is an isomorphism $C_{\mathcal{F}}(U) \to C_{\mathcal{F}}(V)$.* $\square$

Definition 1.6. Let T be a subgroup of S. Then T is *weakly closed* in $\mathcal{F}$ if $T^{\mathcal{F}} = \{T\}$, *strongly closed* in $\mathcal{F}$ if $X^{\mathcal{F}}$ is a set of subgroups of T for each subgroup X of T, and *normal* in $\mathcal{F}$ if $\mathcal{F} = N_{\mathcal{F}}(T)$.

The following result is immediate from the definitions.

Lemma 1.7. *If U and V are subgroups of S which are normal in $\mathcal{F}$ then also UV is normal in $\mathcal{F}$. Thus, there is a largest subgroup $O_p(\mathcal{F})$ of S which is normal in $\mathcal{F}$.* $\square$

A set Δ of subgroups of S is *$\mathcal{F}$-invariant* if $X \in \Delta \implies X^{\mathcal{F}} \subseteq \Delta$. An $\mathcal{F}$-invariant set Δ of subgroups of S is *$\mathcal{F}$-closed* if Δ is non-empty and is closed with respect to overgroups in S ($P \in \Delta$ and $P \leq Q \leq S \implies Q \in \Delta$).

Definition 1.8. Let $\mathcal{F}$ be a fusion system on S, and let $P \leq S$ be a subgroup of S.

(1) P is *centric* in $\mathcal{F}$ (or P is *$\mathcal{F}$-centric*) if $C_S(Q) \leq Q$ for all $Q \in P^{\mathcal{F}}$.
(2) P is *radical* in $\mathcal{F}$ (or P is *$\mathcal{F}$-radical*) if there exists $Q \in P^{\mathcal{F}}$ such that Q is fully normalized in $\mathcal{F}$ and such that $Q = O_p(N_{\mathcal{F}}(Q))$.
(3) P is *quasicentric* in $\mathcal{F}$ (or P is *$\mathcal{F}$-quasicentric*) if there exists $Q \in P^{\mathcal{F}}$ such that Q is fully centralized in $\mathcal{F}$ and such that $C_{\mathcal{F}}(Q)$ is the trivial fusion system on $C_S(Q)$.
(4) P is *subcentric* in $\mathcal{F}$ (or P is *$\mathcal{F}$-subcentric*) if there exists $Q \in P^{\mathcal{F}}$ such that Q is fully normalized in $\mathcal{F}$ and such that $O_p(N_{\mathcal{F}}(Q))$ is centric in $\mathcal{F}$.

Write $\mathcal{F}^c$, $\mathcal{F}^q$, and $\mathcal{F}^s$, respectively, for the set of subgroups of S which are $\mathcal{F}$-centric, $\mathcal{F}$-quasicentric, and $\mathcal{F}$-subcentric. Write $\mathcal{F}^{cr}$ for the set of subgroups of S which are both centric and radical in $\mathcal{F}$.

Remark. The above definition of $\mathcal{F}$-radical subgroup is different from the standard one (which is that P is $\mathcal{F}$-radical if $Inn(P) = O_p(Aut_{\mathcal{F}}(P))$). But it will turn out to be equivalent to the standard definition in the case that $\mathcal{F}$ is the fusion system of a proper locality.

Lemma 1.9. *Let $\mathcal{F}$ be a fusion system on S. Then $\mathcal{F}^{cr}$ is $\mathcal{F}$-invariant, and $\mathcal{F}^c$ is $\mathcal{F}$-closed.*

Proof. Both $\mathcal{F}^c$ and $\mathcal{F}^{cr}$ are $\mathcal{F}$-invariant by definition. Let $P \in \mathcal{F}^c$, let $P \leq Q \leq S$, and let $\phi : Q \to S$ be an $\mathcal{F}$-homomorphism. Then $C_S(Q\phi) \leq C_S(P\phi) \leq P\phi \leq Q\phi$, and so $Q \in \mathcal{F}^c$. Thus $\mathcal{F}^c$ is closed with respect to overgroups in S. As $S \in \mathcal{F}^c$ it follows that $\mathcal{F}^c$ is $\mathcal{F}$-closed. $\square$

Lemma 1.10. *Let $P \leq S$ be a subgroup of S and let $Q \in P^{\mathcal{F}}$ such that Q is fully centralized in $\mathcal{F}$. Then $P \in \mathcal{F}^c$ if and only if $C_S(Q) \leq Q$.*

Proof. Suppose that $C_S(Q) \leq Q$ and let $R \in Q^{\mathcal{F}}$. Then

$$|C_S(R)| \leq |C_S(Q)| = |Z(Q)| = |Z(R)|,$$

and so $C_S(R) = Z(R)$. That is, $C_S(R) \leq R$, and thus Q is $\mathcal{F}$-centric. As $\mathcal{F}^c$ is $\mathcal{F}$-invariant by 1.9, P is then $\mathcal{F}$-centric. That is:

$$C_S(Q) \leq Q \implies P \in \mathcal{F}^c.$$

The reverse implication is given by the definition of $\mathcal{F}^c$. $\square$

Let Ψ be a non-empty set of $\mathcal{F}$-isomorphisms. Then $\mathcal{F}$ is *generated* by Ψ if every $\mathcal{F}$-isomorphism can be expressed as a composition of restrictions of members of Ψ. We may write $\mathcal{F} = \langle \Psi \rangle$ in that case. An important special case is that in which

$$(*) \qquad \Psi = \langle \bigcup \{Aut_{\mathcal{F}}(R) \mid R \in \mathcal{F}^{cr}\rangle.$$

We say that $\mathcal{F}$ is (cr)*-generated* if (*) holds.

Definition 1.11. Let $\mathcal{F}$ be a fusion system on S, and let Γ be an $\mathcal{F}$-closed set of subgroups of S. Then $\mathcal{F}$ is Γ*-inductive* if:

(*) For each $U \in \Gamma$, and each $V \in U^{\mathcal{F}}$ such that V is fully normalized in $\mathcal{F}$, there exists an $\mathcal{F}$-homomorphism $\phi : N_S(U) \to N_S(V)$ with $U\phi = V$.

If $\mathcal{F}$ is Γ-inductive where Γ is the set of all subgroups of S, we shall simply say that $\mathcal{F}$ is *inductive.*

Lemma 1.12. *Let G be a finite group, let S be a Sylow p-subgroup of G, and set $\mathcal{F} = \mathcal{F}_S(G)$. Then $\mathcal{F}$ is inductive, and $\mathcal{F}$ is (cr)-generated.*

Proof. Let V be fully normalized in $\mathcal{F}$. Equivalently: $N_S(V) \in Syl_p(N_G(V))$. Let $U \in V^{\mathcal{F}}$, and let $g \in \mathcal{L}$ with $U^g = V$. Then $N_S(U)^g \leq N_G(V)$, and there then exists $h \in N_G(V)$ with $N_S(U)^{gh} \leq N_S(V)$. Thus $\mathcal{F}$ is inductive. That $\mathcal{F}$ is (cr)-generated is a well-known consequence of the Alperin-Goldschidt fusion theorem [Gold]. $\square$

The next few results provide information about inductive fusion systems. All of these results can be re-stated (and proved) in an obvious way, so as to yield corresponding results about Γ-inductive fusion systems.

Lemma 1.13. *Assume that $\mathcal{F}$ is inductive, and let $V \leq S$ be fully normalized in $\mathcal{F}$. Then V is fully centralized in $\mathcal{F}$.*

Proof. Let $U \in V^{\mathcal{F}}$, and let $\phi : N_S(U) \to N_S(V)$ be an $\mathcal{F}$-homomorphism such that $V = U\phi$. Then ϕ maps $C_S(U)$ into $C_S(V)$, and thus $|C_S(U)| \leq |C_S(V)|$. $\square$

Lemma 1.14. *Assume that $\mathcal{F}$ is inductive, and let U and V be $\mathcal{F}$-conjugate subgroups of S.*

(a) *If U and V are fully normalized in $\mathcal{F}$ then $N_{\mathcal{F}}(U) \cong N_{\mathcal{F}}(V)$.*
(b) *If U and V are fully centralized in $\mathcal{F}$ then $C_{\mathcal{F}}(U) \cong C_{\mathcal{F}}(V)$.*

Proof. Suppose that U and V are fully normalized in $\mathcal{F}$. Then (FL1) implies that there exisits an $\mathcal{F}$-isomorphism $\phi : N_S(U) \to N_S(V)$ with $U\phi = V$. By 1.5, ϕ is then an isomorphism $N_{\mathcal{F}}(U) \to N_{\mathcal{F}}(V)$. Thus (a) holds. Now suppose instead that U and V are fully centralized in $\mathcal{F}$, and let X be a fully normalized $\mathcal{F}$-conjugate of U (and hence also of V). There are then $\mathcal{F}$-homomorphisms $\rho : N_S(U) \to N_S(X)$ and $\sigma : N_S(V) \to N_S(X)$ with $U\rho = X = V\sigma$. The restriction of ρ to $C_S(U)U$ is then an isomorphism with $C_S(X)X$, and similarly σ restricts to an isomorphism $C_S(V)V \to C_S(X)X$. A further application of 1.5 now yields (b). $\square$

Lemma 1.15. *Assume that $\mathcal{F}$ is inductive, let T be strongly closed in $\mathcal{F}$, let $Q \leq S$ be a subgroup of S, and set $V = Q \cap T$. Suppose that V is fully normalized in $\mathcal{F}$ and that Q is fully normalized in $N_{\mathcal{F}}(V)$. Then Q is fully normalized in $\mathcal{F}$.*

Proof. Let $P \in Q^{\mathcal{F}}$ such that P is fully normalized in $\mathcal{F}$, let $\phi : N_S(Q) \to N_S(P)$ be an $\mathcal{F}$-homomorphism with $Q\phi = P$, and set $U = V\phi$. Then $U = P \cap T$ as T is strongly closed in $\mathcal{F}$, and then also $N_S(P) \leq N_S(U)$. By (FL1) there exists an $\mathcal{F}$-homomorphism $\psi : N_S(U) \to N_S(V)$ with $U\psi = V$. Set $Q' = P\psi$. Then $Q' = (Q\phi)\psi$ is an $N_{\mathcal{F}}(V)$-conjugate of Q. Since $N_S(Q) = N_{N_S(V)}(Q)$ (and similarly for Q'), and since Q is fully normalized in $N_{\mathcal{F}}(V)$, we have $|N_S(Q)| \geq |N_S(Q')|$. The sequence

$$N_S(Q) \xrightarrow{\phi} N_S(P) \xrightarrow{\psi} N_S(Q')$$

of injective homomorphisms then shows that $|N_S(P)| = |N_S(Q)|$, and so Q is fully normalized in $\mathcal{F}$. $\square$

Lemma 1.16. *Assume that $\mathcal{F}$ is inductive. Let $V \leq S$ be fully normalized in $\mathcal{F}$, and let Q be fully centralized in $N_{\mathcal{F}}(V)$. Suppose that $V \leq Q$. Then Q is fully centralized in $\mathcal{F}$.*

Proof. As in the proof of 1.15: Let $P \in Q^{\mathcal{F}}$ such that P is fully normalized in $\mathcal{F}$, and let $\phi : N_S(Q) \to N_S(P)$ be an $\mathcal{F}$-homomorphism with $Q\phi = P$. Set $U = V\phi$ and let $\psi : N_S(U) \to N_S(V)$ be an $\mathcal{F}$-homomorphism with $U\psi = V$. Set $Q' = P\psi$. Then Q' is an $N_{\mathcal{F}}(V)$-conjugate of Q. As Q is fully centralized in $N_{\mathcal{F}}(V)$, and since $C_S(Q) = C_{C_S(V)}(Q)$ (and similarly for Q'), we then have $|C_S(Q)| \geq |C_S(Q')|$. The sequence

$$C_S(Q) \xrightarrow{\phi} C_S(P) \xrightarrow{\psi} C_S(Q')$$

then shows that $|C_S(P)| = |C_S(Q)|$. On the other hand, as P is fully normalized in $\mathcal{F}$, P is also fully centralized in $\mathcal{F}$ by 1.13. Thus $|C_S(P)| \geq |C_S(Q)|$, and the lemma follows. $\square$

Lemma 1.17. *Assume that $\mathcal{F}$ is inductive, let $T \leq S$ be strongly closed in $\mathcal{F}$, and let $\mathcal{E}$ be a fusion subsystem of $\mathcal{F}$ on T. Let $U \leq T$ be a subgroup of T such that U is fully normalized in $\mathcal{F}$. Then U is fully normalized in $\mathcal{E}$.*

Proof. Let $V \in U^{\mathcal{E}}$ such that V is fully normalized in $\mathcal{E}$. Then $V \in U^{\mathcal{F}}$. As $\mathcal{F}$ is inductive there exists an $\mathcal{F}$-homomorphism $\phi : N_S(V) \to N_S(U)$ with $V\phi = U$. Then $N_T(V)\phi \leq N_T(U)$ as T is strongly closed in $\mathcal{F}$. As V is fully normalized in $\mathcal{E}$ it follows that $N_T(V)\phi = N_T(U)$, and hence U is fully normalized in $\mathcal{E}$. $\square$

Lemma 1.18. *Assume that $\mathcal{F}$ is inductive, and let $U \leq S$ be a subgroup of S. Then there exists $V \in U^{\mathcal{F}}$ such that both V and $O_p(N_{\mathcal{F}}(V))$ are fully normalized in $\mathcal{F}$.*

Proof. Without loss of generality, U is fully normalized in $\mathcal{F}$. Set $P = O_p(N_{\mathcal{F}}(U))$ and let $Q \in P^{\mathcal{F}}$ with Q fully normalized in $\mathcal{F}$. Let $\phi : N_S(P) \to N_S(Q)$ be an $\mathcal{F}$-homomorphism which maps P to Q, and set $V = U\phi$. As $N_S(U) \leq N_S(P)$ we have $N_S(U)\phi \leq N_S(V)$. But $|N_S(U)| \geq |N_S(V)|$ as U is fully normalized, so ϕ induces an isomorphism $N_S(U) \to N_S(V)$. Thus V is fully normalized in $\mathcal{F}$, and then $Q = O_p(N_{\mathcal{F}}(V))$ by 1.14(a). $\square$

We end with this section with a basic result on quotients of fusion systems.

Lemma 1.19. *Let $\mathcal{F}$ be a fusion system on S, let $\overline{\mathcal{F}}$ be a fusion system on $\overline{S}$, and let $\lambda : S \to \overline{S}$ be a fusion-preserving homomomorphism. Denote also by λ the corresponding homomorphism $\mathcal{F} \to \overline{\mathcal{F}}$ of fusion systems (cf. 1.3). Assume that λ is surjective, and that each of the mappings*

$$\lambda_{X,Y} : Hom_{\mathcal{F}}(X, Y) \to Hom_{\overline{\mathcal{F}}}(X\lambda, Y\lambda) \quad (Ker(\lambda) \leq X, Y \leq S)$$

is surjective. Let $P \leq S$ with $Ker(\lambda) \leq P$. Then the following hold.

(a) *P is fully normalized in $\mathcal{F}$ if and only if $\overline{P}$ is fully normalized in $\overline{\mathcal{F}}$.*
(b) *$O_p(N_{\mathcal{F}}(P))\lambda \leq O_p(N_{\overline{\mathcal{F}}}(\overline{P}))$.*
(c) *If $P\lambda \in \overline{\mathcal{F}}^c$ then $P \in \mathcal{F}^c$, and if $P\lambda \in \overline{\mathcal{F}}^{cr}$ then $P \in \mathcal{F}^{cr}$.*

Proof. For any subgroup U of S containing $Ker(\lambda)$ write $\overline{U}$ for $U\lambda$. There is then no ambiguity in saying that for any subgroup or element $\overline{U}$ of $\overline{S}$ we shall write U for the preimage of $\overline{U}$ in S. For any such U we have $\overline{N_S(U)} = N_{\overline{S}}(\overline{U})$, and it is this observation that yields (a).

Set $Q = O_p(N_{\mathcal{F}}(P))$ and set $\overline{R} = O_p(N_{\overline{\mathcal{F}}}(\overline{P})$. Let $\overline{\phi} : \overline{X} \to \overline{Y}$ be a $N_{\overline{\mathcal{F}}}(\overline{P})$-homomorphism. By hypothesis there exists an $N_{\mathcal{F}}(P)$-homomorphism $\phi : X \to Y$ with $\overline{\phi} = (\phi)\lambda$. Then ϕ extends to an $N_{\mathcal{F}}(P)$-homomorphism $\psi : QX \to QY$ which fixes Q, and then $(\psi)\lambda$ is an extension of $\overline{\phi}$ to an $N_{\overline{\mathcal{F}}}(\overline{P})$-homomorphism $\overline{QX} \to \overline{QY}$ which fixes $\overline{Q}$. This shows that $\overline{Q} \trianglelefteq N_{\overline{\mathcal{F}}}(\overline{P})$, and so $\overline{Q} \leq \overline{R}$. A similar argument - whose details may safely be omitted - shows that $R \leq Q$, and establishes (b).

The first statement in (c) is immediate from the observation that $\overline{C_S(U)} \leq C_{\overline{S}}(\overline{U})$ for any subgroup $U \leq S$. For any fusion system $\mathcal{E}$ on a p-group T write $\mathcal{E}^r$ for the set of $\mathcal{E}$-radical subgroups of T (cf. 1.8). Suppose that $\overline{P} \in \overline{\mathcal{F}}^r$. By definition 1.8 there is then an $N_{\overline{\mathcal{F}}}(\overline{P})$-conjugate $\overline{P}_1$ of $\overline{P}$ such that $\overline{P}_1$ is fully normalized in $\overline{\mathcal{F}}$ and such that $\overline{P}_1 = O_p(N_{\overline{\mathcal{F}}}(\overline{P}_1))$. Then P_1 is an $\mathcal{F}$-conjugate of P, P_1 is fully normalized in $\mathcal{F}$ by (a), and $P_1 = O_p(N_{\mathcal{F}}(P_1))$ by (b). Thus $P \in \mathcal{F}^r$, and (c) holds. $\square$

Section 2: Fusion systems of localities

Throughout this section, fix a locality $(\mathcal{L}, \Delta, S)$. Define $\mathcal{F}_S(\mathcal{L})$ to be the smallest fusion system $\mathcal{F}$ on S which contains the homomorphisms $c_g : S_g \to S$, where c_g denotes conjugation by $g \in \mathcal{L}$. Equivalently, for each pair of subgroups U, V of S define $Hom_{\mathcal{F}}(U, V)$ to be the set of mappings

$$c_w : U \to V,$$

where $w = (g_1, \cdots, g_n) \in \mathbf{W}(\mathcal{L})$, $U \leq S_w$, and where the composition

$$c_w = c_{g_1} \circ \cdots \circ c_{g_n}$$

of conjugation maps carries U into V. We say that $\mathcal{L}$ is a locality *on* $\mathcal{F}$.

Lemma 2.1. *Let $(\mathcal{L}, \Delta, S)$ be a locality on $\mathcal{F}$, and let $P \in \Delta$. Then*

$$(*) \qquad P^{\mathcal{F}} = \{P^g \mid g \in \mathcal{L},\ P \leq S_g\}.$$

Moreover, P is fully normalized in $\mathcal{F}$ if and only if $N_S(P) \in Syl_p(N_{\mathcal{L}}(P))$, and P is fully centralized in $\mathcal{F}$ if and only if $C_S(P) \in Syl_p(C_{\mathcal{L}}(P))$.

Proof. Let $\phi : P \to Q$ be an $\mathcal{F}$-isomorphism. As noted above, $\phi = c_w$ for some $w \in \mathbf{W}(\mathcal{L})$ with $P \leq S_w$. Then $w \in \mathbf{D}$, and $c_w = c_{\Pi(w)}$ by I.2.3(c). This yields (*).

As P is in Δ, $N_{\mathcal{L}}(P)$ is a subgroup of $\mathcal{L}$, and $C_{\mathcal{L}}(P)$ is a normal subgroup of $N_{\mathcal{L}}(P)$. Let X be a Sylow p-subgroup of $N_{\mathcal{L}}(P)$ containing $N_S(P)$. By I.2.11 there exists $g \in \mathcal{L}$ with $X^g \leq S$, and conjugation by g induces an isomorphism $N_{\mathcal{L}}(P) \to N_{\mathcal{L}}(P^g)$ by I.2.3(b). If

P is fully normalized in $\mathcal{F}$ then $|N_S(P)| \geq |N_S(P^g)| \geq |X^g| = |X|$, so $N_S(P) = X$, and $N_S(P) \in Syl_p(N_{\mathcal{L}}(P)$ in that case. Conversely, suppose that $N_S(P)$ is a Sylow subgroup of $N_{\mathcal{L}}(P)$ and let $Q \in P^{\mathcal{F}}$. We have $P = Q^h$ for some $h \in \mathcal{L}$ by (*), so $N_S(Q)^h \leq N_{\mathcal{L}}(P)$. By Sylow's theorem there exists $f \in N_{\mathcal{L}}(P)$ with $(N_S(Q)^g)^f \leq N_S(P)$, and thus $|N_S(P)| \geq |N_S(Q)|$. This establishes the first of the two "if and only ifs" of the lemma. The proof of the second "if and only if" is obtained in similar fashion. □

Proposition 2.2. *Let $(\mathcal{L}, \Delta, S)$ be a locality, and set $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$. Then $\mathcal{F}$ is Δ-inductive. Moreover, for each $P \in \Delta$ such that P is fully normalized in $\mathcal{F}$:*

(a) *$N_{\mathcal{F}}(P) = \mathcal{F}_{N_S(P)}(N_{\mathcal{L}}(P))$ and $C_{\mathcal{F}}(P) = \mathcal{F}_{C_S(P)}(C_{\mathcal{L}}(P))$.*
(b) *$N_{\mathcal{F}}(P)$ and $C_{\mathcal{F}}(P)$ are (cr)-generated.*

Proof. That $\mathcal{F}$ is Δ-inductive is immediate from the preceding lemma and from I.2.10. Let $P \in \Delta$ with P fully normalized in $\mathcal{F}$, and let $\phi : X \to Y$ be an $N_{\mathcal{F}}(P)$-isomorphism between two subgroups X and Y of $N_S(P)$ containing P. As in the proof of (*) in 2.1, we find that $\phi = c_g$ for some $g \in N_{\mathcal{L}}(P)$, and this shows that $N_{\mathcal{F}}(P) = \mathcal{F}_{N_S(P)}(N_{\mathcal{L}}(P))$. Then $N_{\mathcal{F}}(P)$ is $N_{\mathcal{F}}(P)^{cr}$-generated by 1.12.

By 1.13 P is fully centralized in $\mathcal{F}$. Let $\phi : X \to Y$ be a $C_{\mathcal{F}}(P)$-isomorphism between two subgroups X and Y of $C_S(P)$. By definition of $C_{\mathcal{F}}(P)$, ϕ extends to an $\mathcal{F}$-isomorphism $\psi : XP \to YP$ such that ψ restricts to the identity map on P. Then $\psi = c_g$ for some $g \in C_{\mathcal{L}}(P)$, and thus $C_{\mathcal{F}}(P) = \mathcal{F}_{C_S(P)}(C_{\mathcal{L}}(P))$. We again appeal to 1.12, obtaining (cr)-generation for $C_{\mathcal{F}}(P)$. □

A finite group G is of *characteristic p* if $C_G(O_p(G)) \leq O_p(G)$.

Lemma 2.3. *Let $(\mathcal{L}, \Delta, S)$ be a locality on $\mathcal{F}$, and assume that $N_{\mathcal{L}}(P)$ is of characteristic p for all $P \in \Delta$. Then $O_p(\mathcal{F}) = O_p(\mathcal{L})$.*

Proof. As $\mathcal{F}$ is generated by the conjugation maps $c_g : S_g \to S$ for $g \in \mathcal{L}$, the inclusion $O_p(\mathcal{L}) \leq O_p(\mathcal{F})$ is immediate.

Set $R = O_p(\mathcal{F})$ and assume by way of contradiction that R is not normal in $\mathcal{L}$. Among all elements of $\mathcal{L}$ not in $N_{\mathcal{L}}(R)$, choose g so that $|S_g|$ is as large as possible. Then $R \not\leq S_g$, since $c_g : S_g \to S$ is an $\mathcal{F}$-homomorphism. In particular, $S_g \neq S$, and S_g is a proper subgroup of $N_S(S_g)$.

Set $P = S_g$, $P' = P^g$, and let $Q \in P^{\mathcal{F}}$ be fully normalized in $\mathcal{F}$. As $\mathcal{L}$ is Δ-inductive by 2.2, there exists $x \in \mathcal{L}$ such that $P^x = Q$ and such that $N_S(P) \leq S_x$. Then also $R \leq S_x$ by the maximality of $|P|$ in the choice of g. Since $Q \in (P')^{\mathcal{F}}$ there exists also $y \in \mathcal{L}$ such that $(P')^y = Q$ and such that $N_S(P')R \leq S_y$.

Note that $(x^{-1}, g, y) \in \mathbf{D}$ via Q, and that $f := \Pi(x^{-1}, g, y) \in N_{\mathcal{L}}(Q)$. Note also that $(x, x^{-1}, g, y, y^{-1}) \in \mathbf{D}$ via P, and that

$$\Pi(x, f, y^{-1}) = \Pi(x, x^{-1}, g, y, y^{-1}) = \Pi(g)$$

by $\mathbf{D}$-associativity (I.1.4). If $R \leq S_f$ then $R \leq S_{(x,f,y^{-1})}$, and then $R \leq S_g$. Thus $R \not\leq S_f$, and we may therefore replace g with f, and P with Q. That is, we may assume that P is fully normalized in $\mathcal{F}$ and $g \in N_{\mathcal{L}}(P)$.

Set $M = N_{\mathcal{L}}(P)$. Then M is a subgroup of $\mathcal{L}$ and, by hypothesis, M is of characteristic p. Set $D = N_R(P)$, and set $\mathcal{E} = \mathcal{F}_{N_S(P)}(M)$. Then $D \trianglelefteq \mathcal{E}$, and so the conjugation map $c_g : P \to P$ extends to an $\mathcal{E}$-automorphism of PD. Thus, there exists $h \in N_M(PD)$ such that $c_h : PD \to PD$ restricts to c_g on P. Then $gh^{-1} \in C_M(P) \leq P$, and so $gh^{-1} \in N_M(D)$. This yields $D^g = D$, so $D \leq P$, and then $R \leq P$. Then $R^g = R$. This result is contrary to the choice of g, and completes the proof. $\square$

Definition 2.4. Let $(\mathcal{L}, \Delta, S)$ be a locality on $\mathcal{F}$. Then $\mathcal{L}$ is *proper* if:

(PL1) $\mathcal{F}^{cr} \subseteq \Delta$, and

(PL2) $N_{\mathcal{L}}(P)$ is of characteristic p for each $P \in \Delta$.

The next two results are well known, and are important for an understanding of the structure of finite groups of characteristic p.

Lemma 2.5. *Let G be a finite group, and let A and B be subgroups of G such that $|A|$ is relatively prime to $|B|$. Suppose that $[A, B] \leq C_B(A)$. Then $[A, B] = 1$.*

Proof. Let $a \in A$ and $b \in B$. Then $a^{-1}a^b = [a, b]$ commutes with a, by hypothesis, and so a^b commutes with $\langle a \rangle$. As $|a| = |a^b|$ is relatively prime to $|B|$, the same is then true of $a^{-1}a^b$. As $[a, b] \in B$ by hypothesis, we conclude that $[a, b] = 1$, and thus $[A, B] = 1$. $\square$

Lemma 2.6 (Thompson's $A \times P$ Lemma). *Let G be a finite group and let $H \leq G$ be a subgroup of G such that:*

(i) *$H = PA$, where P is a normal p-subgroup of H, and where A is a p'-subgroup of H.*

(ii) *There exists a subgroup B of $C_P(A)$ such that $[C_P(B), A] = 1$.*

Then $[P, A] = 1$ and H is isomorphic to the direct product $A \times P$.

Proof. We have $A \cap P = 1$ since $|A|$ and $|P|$ are relatively prime, and so it suffices to show that $P = C_P(A)$. Suppose false, so that $C_P(A)$ is a proper subgroup of P. Set $Q = N_P(C_P(A))$. Thus $B \leq C_P(A) \trianglelefteq Q$, and so $[Q, B] \leq C_P(A)$. One may express this in the standard way as $[Q, B, A] = 1$. Also $[B, A, Q] = 1$ since $B \leq C_P(A)$. The Three Subgroups Lemma (2.2.3 in [Gor]) then yields $[A, Q, B] = 1$. That is, we have $[A, Q] \leq C_Q(B)$, and hence $[A, Q] \leq C_Q(A)$ by (ii). Then $Q \leq C_P(A)$ by 2.5. That is, $N_P(C_P(A)) \leq C_P(A)$, and hence $P = C_P(A)$, as required. $\square$

Lemma 2.7. *Let G be a finite group of characteristic p.*

(a) *Every normal subgroup of G is of characteristic p.*

(b) *Every p-local subgroup of G is of characteristic p.*

(c) *Let V be a normal p-subgroup of G, and let X be the set of elements $x \in C_G(V)$ such that $[O_p(G), x] \leq V$. Then X is a normal p-subgroup of G.*

Proof. Let $K \trianglelefteq G$ be a normal subgroup of G, and set $R = O_p(K)$. Then $[O_p(G), K] \leq R$, and so $[O_p(G), C_K(R), C_K(R)] = 1$. Then $O^p(C_K(R)) \leq C_G(O_p(G))$ by 2.5, and thus $O^p(C_K(R)) \leq O_p(G)$. Then $C_K(R)$ is a normal p-subgroup of K, and so $C_K(R) \leq R$. This establishes point (a).

Next, let U be a p-subgroup of G, and set $H = N_G(U)$, $P = O_p(H)$, and $Q = O_p(G)$. Then $N_Q(U) \leq P$. Let A be a p'-subgroup of $C_H(P)$. Then $[N_Q(U), A] = 1$, and so $[Q, A] = 1$ by 2.6. Then $A \leq Q$, and thus $A = 1$, proving (b).

For the proof of (c), notice that $C_G(V) \trianglelefteq G$, and that X is the intersection of $C_G(V)$ with the preimage in G of the normal subgroup $C_{G/V}(O_p(G)/V)$ of G/V. Thus $X \trianglelefteq G$. Each p'-subgroup of X centralizes $O_p(G)$ by 2.5, so X is a p-group. $\square$

The next result refers to the terminology and notation of 1.8.

Lemma 2.8. *Let $(\mathcal{L}, \Delta, S)$ be a proper locality on $\mathcal{F}$ and let $P \in \Delta$. Then P is subcentric in $\mathcal{F}$, and the following hold.*

(a) *$P \in \mathcal{F}^{cr}$ if and only if $P = O_p(N_{\mathcal{L}}(P))$.*
(b) *P is centric in $\mathcal{F}$ if and only if $C_{\mathcal{L}}(P) = Z(P)$.*
(c) *P is quasicentric in $\mathcal{F}$ if and only if $C_{\mathcal{L}}(P) \leq O_p(N_{\mathcal{L}}(P))$.*

Proof. Set $M = N_{\mathcal{L}}(P)$, and let $Q \in P^{\mathcal{F}}$ with Q fully normalized in $\mathcal{F}$. Then $Q = P^g$ for some $g \in \mathcal{L}$ by 2.1, and then the conjugation map $c_g : M \to M^g$ is an isomorphism by I.2.3(b). As $\mathcal{F}^{cr}$, $\mathcal{F}^c$, $\mathcal{F}^q$, and $\mathcal{F}^s$ are $\mathcal{F}$-invariant by 1.9, it follows that it suffices to establish the lemma under the assumption (which we now make) that P itself is fully normalized in $\mathcal{F}$. Since M may be regarded as a proper locality whose set of objects is the set of all subgroups of $N_S(P)$, it follows from 2.2 and 2.3 that $O_p(N_{\mathcal{F}}(P)) = O_p(M)$. As $\mathcal{F}$ is Δ-inductive by 2.3, and M is of characteristic p, it follows from 1.10 and 1.13 that $O_p(M)$ is centric in $\mathcal{F}$. Thus $P \in \mathcal{F}^s$.

Set $K = C_{\mathcal{L}}(P)$. As $\mathcal{L}$ is proper, M is of characteristic p, and then K is of characteristic p by 2.7(a). Further, we have $C_{\mathcal{F}}(P) = \mathcal{F}_{C_S(P)}(K)$ by 2.2. If $P \in \mathcal{F}^c$ then $C_S(P) = Z(P) \leq Z(K)$, so $O^p(K)$ is a normal p'-subgroup of K. Then $O^p(K) = 1$, and $K = Z(P)$. Conversely, if $K = Z(P)$ then $C_S(P) \leq P$. As P is fully normalized in $\mathcal{F}$, P is also fully centralized by 1.13, and we then conclude from 1.10 that $P \in \mathcal{F}^c$. This establishes (b).

Suppose next that $P \in \mathcal{F}^q$, so that $C_{\mathcal{F}}(P)$ is the trivial fusion system on $C_S(P)$. Then $N_K(U)/C_K(U)$ is a p-group for every subgroup U of $C_S(P)$. Take $U = O_p(K)$. Thus $K/Z(U)$ is a p-group, so K is a p-group, and then $K = C_S(P)$ is a normal p-subgroup of M. Conversely, if $K \leq O_p(M)$ then $C_{\mathcal{F}}(P)$ is trivial, so we have (c).

Set $R = O_p(M)$ and let Γ be the set of all overgroups of R in $N_S(P)$. As M is of characteristic p, R is centric in $N_{\mathcal{F}}(P)$, and then $\Gamma \subseteq N_{\mathcal{F}}(P)^c$. We may view M as a locality $(M, \Gamma, N_S(P))$ which happens to be a group, and this locality is proper by 2.7(b). Then $R = O_p(N_{\mathcal{F}}(P))$ by 2.3. This completes the proof of (a). $\square$

The next result shows that if $(\mathcal{L}, \Delta, S)$ is a locality on $\mathcal{F}$, such that the set Δ of objects is not "too small" ($\mathcal{F}^{cr} \subseteq \Delta$) and not "too large" ($\Delta \subseteq \mathcal{F}^q$), then $\mathcal{L}$ has a canonical homomorphic image which is a proper locality on $\mathcal{F}$.

Proposition 2.9. *Let $(\mathcal{L}, \Delta, S)$ be a locality on $\mathcal{F}$, with $\mathcal{F}^{cr} \subseteq \Delta \subseteq \mathcal{F}^q$. Let Θ be the union of the groups $\Theta(P) := O_{p'}(C_{\mathcal{L}}(P))$ over all $P \in \Delta$. Then Θ is a partial normal subgroup of $\mathcal{L}$, and $\mathcal{L}/\Theta$ is a proper locality on $\mathcal{F}$.*

Proof. Let $P \in \Delta$ and let $Q \in P^{\mathcal{F}}$ be fully normalized in $\mathcal{F}$. Set $M = N_{\mathcal{L}}(Q)$ and $K = C_{\mathcal{L}}(Q)$. Then $C_{\mathcal{F}}(Q) = \mathcal{F}_{C_S(Q)}(K)$ by 2.2. As Q is $\mathcal{F}$-quasicentric by hypothesis,

$C_{\mathcal{F}}(Q)^{cr}$ is the trivial fusion system on $C_S(Q)$, and a classical theorem of Frobenius [Theorem 7.4.5 in Gor] then implies that K has a normal p-complement. That is, we have $K = \Theta(Q)C_S(Q)$.

Set $\overline{M} = M/\Theta(Q)$, set $\overline{R} = O_p(\overline{M})$, and let R be the preimage of $\overline{R}$ in $N_S(Q)$. Let $\overline{X}$ be a p'-subgroup of $C_{\overline{M}}(\overline{R})$, and let X be the preimage of $\overline{X}$ in M. Since $M = N_M(R)\Theta(Q)$ by the Frattini argument, we obtain $X = N_X(R)\Theta(Q)$. But

$$[R, N_X(R)] \leq R \cap [R, X] \leq R \cap \Theta(Q) = 1,$$

so

$$N_X(R) \leq C_X(R) \leq O^p(C_K(Q)) = \Theta(Q),$$

and thus $X = \Theta(Q)$. This shows that $C_{\overline{M}}(\overline{R})$ is a p-group, and thus $\overline{M}$ is of characteristic p. In this way the hypothesis (*) of I.4.12 is fulfilled, and we conclude that $\Theta \trianglelefteq \mathcal{L}$, $(\mathcal{L}/\Theta, \Delta, S)$ is a locality, $\mathcal{F} = \mathcal{F}_S(\mathcal{L}/\Theta)$, and $N_{\mathcal{L}/\Theta}(P)$ is of characteristic p for all $P \in \Delta$. As $\mathcal{F}^{cr} \subseteq \Delta$, $\mathcal{L}/\Theta$ is a proper locality on $\mathcal{F}$. $\square$

There are proper localities $(\mathcal{L}, \Delta, S)$, with fusion system $\mathcal{F}$, such that Δ is strictly larger than $\mathcal{F}^q$. For example, if G is a finite group of Lie type, defined over a field of characteristic p, and Δ is the set of all non-identity subgroups of a Sylow p-subgroup S of G, then a theorem of Borel and Tits shows that $N_G(P)$ is of characteristic p for all $P \in \Delta$. Then $(G \mid_\Delta, \Delta, S)$ is a proper locality on $\mathcal{F}_S(G)$, whereas in general there are non-identity subgroups of S which are not quasicentric in $\mathcal{F}$.

Lemma 2.10. *Let $(\mathcal{L}, \Delta, S)$ be a proper locality on $\mathcal{F}$. Then $\mathcal{F}$ is (cr)-generated.*

Proof. Let Γ be the set of all $R \in \mathcal{F}^{cr}$ and such that R is fully normalized in $\mathcal{F}$, and let $\mathcal{F}_0$ be the fusion system on S generated by the union of the groups $Aut_{\mathcal{F}}(R)$, for $R \in \Gamma$. Assuming the lemma to be false, there exists an $\mathcal{F}$-isomorphism $\phi : P \to P'$ such that ϕ is not an $\mathcal{F}_0$-isomorphism.

By definition, $\mathcal{F}$ is generated by the conjugation maps $c_g : S_g \to S_{g^{-1}}$, so we may take ϕ to be such a c_g, with $P = S_g$ and $P' = P^g$ (and where P and P' are in Δ). Among all such obstructions g to the lemma, choose g so that $|P|$ is as large as possible. As $S \in \Gamma$, we have $P \neq S$, so P is a proper subgroup of $N_S(P)$.

Let $Q \in P^{\mathcal{F}}$ with Q fully normalized in $\mathcal{F}$. As $\mathcal{F}$ is Δ-inductive by 2.2, there exist elements $x, y \in \mathcal{L}$ such that $N_S(P)^x \leq N_S(Q) \geq N_S(P')^y$, and such that $P^x = Q = (P')^y$. As P is a proper subgroup of $N_S(P)$, c_x and c_y are $\mathcal{F}_0$-isomorphisms. If also $c_x^{-1} \circ c_g \circ c_y$ is an $\mathcal{F}_0$-isomorphism, then so is c_g, as is contrary to the case. We may therefore replace g with $x^{-1}gy$, whence $g \in N_{\mathcal{L}}(Q)$.

Set $Q^* = O_p(N_{\mathcal{F}}(Q))$. Then $Q^* = O_p(N_{\mathcal{L}}(Q))$ by 2.2, and Q^* is $\mathcal{F}$-centric as $\mathcal{L}$ is proper. If $Q = Q^*$ then $Q \in \Gamma$, and then $\phi \in \mathcal{F}_0$. Thus Q is a proper subgroup of Q^*. By the definition of $O_p(N_{\mathcal{F}}(Q))$, ϕ extends to an $\mathcal{F}$-automorphism ϕ^* of Q^*, which is then an $\mathcal{F}_0$-automorphism by the maximality of $|Q|$. Thus ϕ is the restiction of an $\mathcal{F}_0$-automorphism, and so ϕ is in $\mathcal{F}_0$. This contradiction proves the lemma. $\square$

Lemma 2.11. *Let $(\mathcal{L}, \Delta, S)$ be a locality on $\mathcal{F}$, and let Δ_0 be an $\mathcal{F}$-closed subset of Δ. Set*

$$\mathcal{L}_0 = \{g \in \mathcal{L} \mid S_g \in \Delta_0\},$$

and set

$$\mathbf{D}_0 = \{w \in \mathbf{W}(\mathcal{L}_0) \mid S_w \in \Delta_0\},$$

Let $\Pi_0 : \mathbf{D}_0 \to \mathcal{L}_0$ be the restriction of the product Π on $\mathcal{L}$ to Δ_0. Then $\mathcal{L}_0$, together with Π_0 and the restriction to $\mathcal{L}_0$ of the inversion map on $\mathcal{L}$, is a partial group, and $(\mathcal{L}_0, \Delta_0, S)$ is a locality. Moreover, if $\mathcal{L}$ is proper and $\mathcal{F}^{cr} \subseteq \Delta_0$, then $\mathcal{L}_0$ is a proper locality on $\mathcal{F}$.

Proof. The proof that $(\mathcal{L}_0, \Pi_0, (-)^{-1})$ is a partial group is a straightforward exercise with definition I.1.1, and is omitted. Here $\mathbf{D}_0 = \mathbf{D}_{\Delta_0}$, as defined in I.2.1. Since Δ_0 is $\mathcal{F}$-closed, $(\mathcal{L}_0, \Delta_0)$ is then an objective partial group. Every subgroup of $\mathcal{L}_0$ is a subgroup of $\mathcal{L}$, so S is maximal in the poset of p-subgroups of $\mathcal{L}_0$, and thus $(\mathcal{L}_0, \Delta_0, S)$ is a locality. Assume now that $\mathcal{L}$ is proper, and notice that $N_{\mathcal{L}_0}(P) = N_{\mathcal{L}}(P)$ for all $P \in \Delta_0$. Thus, normalizers of objects in $\mathcal{L}_0$ are of characteristic p. Suppose further that $\mathcal{F}^{cr} \subseteq \Delta_0$. Then $\mathcal{F}$ is a subsystem of $\mathcal{F}_S(\mathcal{L}_0)$ by 2.10. The reverse inclusion of fusion systems is obvious, so $\mathcal{F} = \mathcal{F}_S(\mathcal{L}_0)$, and hence $\mathcal{L}_0$ is a proper locality on $\mathcal{F}$. □

The locality $(\mathcal{L}_0, \Delta_0, S)$ in 2.11 will be referred to as the *restriction of* $\mathcal{L}$ *to* Δ_0. It may be denoted $\mathcal{L}\mid_{\Delta_0}$. For example, if G is a finite group, S is a Sylow p-subgroup of G, and Δ is an $\mathcal{F}_S(G)$-closed set of subgroups of S, then one has the restriction $\mathcal{L}_\Delta(G) = G\mid_\Delta$ of G, where G is viewed as a locality whose set of objects is the set of all subgroups of S.

Section 3: Elementary expansions

This section is based closely on [section 5 in Ch1], but without some of the technical complications that were necessary to that earlier paper. Recall the notion of Γ-inductive set from 1.11.

Lemma 3.1. *Let $(\mathcal{L}, \Delta, S)$ be a locality on $\mathcal{F}$, let $R \leq S$ with R fully normalized in $\mathcal{F}$, and suppose that $\langle U, V\rangle \in \Delta$ for every pair $U \neq V$ of distinct $\mathcal{F}$-conjugates of R. Set $\Delta_R = \{P \in \Delta \mid R \trianglelefteq P\}$. Then the following hold.*

(a) *$\Delta_R = \{N_Q(R) \mid R \leq Q \in \Delta\}$, and $(N_{\mathcal{L}}(R), \Delta_R, N_S(R))$ is a locality.*
(b) *$R^{\mathcal{F}} = \{R^g \mid g \in \mathcal{L},\ R \leq S_g\}$.*
(c) *$\mathcal{F}$ is $\Delta \cup R^{\mathcal{F}}$-inductive.*

Proof. Let $Q \in \Delta$ with $R \leq Q$. If $R \trianglelefteq Q$ then $N_Q(R) = Q \in \Delta$, while if $R \ntrianglelefteq Q$ then $N_Q(R)$ contains a pair of distinct Q-conjugates of R. Thus $N_Q(R) \in \Delta$ in any case, and $\Delta_R = \{N_Q(R) \mid R \leq Q \in \Delta\}$.

By I.2.12(a) $N_{\mathcal{L}}(R)$ is a partial subgroup of $\mathcal{L}$. Let X be a p-subgroup of $N_{\mathcal{L}}(R)$ containing $N_S(R)$. By I.2.11 there exists $g \in \mathcal{L}$ with $X^g \leq S$, and then $X^g \leq N_S(R^g)$. As R is fully normalized in $\mathcal{F}$ it follows that $N_S(R) = X$, and then $(N_{\mathcal{L}}(R), \Delta_R, N_S(R))$ is a locality by I.2.12(c). Thus (a) holds.

Let $V \in R^{\mathcal{F}}$, and let $\mathbf{Y}_V$ be the set of elements $y \in \mathcal{L}$ such that $R^y = V$ and such that $N_S(V) \leq S_{y^{-1}}$. By definition, each $\mathcal{F}$-isomorphism $\phi : V \to R$ can be factored as a composition of conjugation maps

$$(*) \qquad V \xrightarrow{c_{x_1}} V_1 \to \cdots \to V_{k-1} \xrightarrow{c_{x_k}} R$$

for some word $w = (x_1, \cdots, x_k) \in \mathbf{W}(\mathcal{L})$. Write $V^w = R$ to indicate this. Assume now that there exists $V \in R^{\mathcal{F}}$ such that $\mathbf{Y}_V = \emptyset$. Among all such V, choose V so that the minimum length k, taken over all words w for which $V^w = R$, is as small as possible. Then, subject to this condition, choose w so that $|N_{S_w}(V)|$ is as large as possible. We evidently have $k > 0$, and $R \neq S$.

Suppose that $k = 1$. Thus $w = (x)$ for some $x \in \mathcal{L}$ with $V^x = R$. Set $P = N_{S_x}(V)$ and set $\widetilde{P} = N_{N_S(V)}(P)$. Then $P \in \Delta$ by 3.1(1). Conjugation by x then induces an isomorphism $N_{\mathcal{L}}(P) \to N_{\mathcal{L}}(P^x)$ by I.2.3(b). Thus $\widetilde{P}^x$ is a p-subgroup of the locality $N_{\mathcal{L}}(R)$. By I.2.11 there exists $z \in N_{\mathcal{L}}(R)$ with $(\widetilde{P}^x)^z \leq N_S(R)$, and then $(x, z) \in \mathbf{D}$ via P. Now $\widetilde{P}^{xz} \leq N_S(R)$, and the maximality of $|N_{S_w}(V)|$ in the choice of V and w yields $P = \widetilde{P}$. Then $P = N_S(V)$, so $x^{-1} \in \mathbf{Y}_V$. This shows that $k > 1$.

The minimality of k now yields $\mathbf{Y}_{V_1} \neq \emptyset$, where V_1 is defined by (*). Thus $V^{(c,d)} = R$, where d may be chosen in $\mathbf{Y}_{V_1}$. Set $A = N_{S_c}(V)$. Then $A \in \Delta$ as $S_c \in \Delta$, and $A^c \leq N_S(V_1)$. As $b \in \mathbf{Y}_{V_1}$ we then have $w \in \mathbf{D}$ via A. Thus $k = 1$, contrary to the result of the preceding paragraph. We conclude that $\mathbf{Y}_V = \emptyset$ for $V \in R^{\mathcal{F}}$, and this proves (b) and (c). $\square$

We assume the following setup for the remainder of this section.

Hypothesis 3.2. $(\mathcal{L}, \Delta, S)$ is a locality on $\mathcal{F}$, and $R \leq S$ is a subgroup of S such that:

(1) Each strict overgroup of R in S is in Δ.
(2) Both R and $O_p(N_{\mathcal{F}}(R))$ are fully normalized in $\mathcal{F}$.
(3) $N_{\mathcal{L}}(R)$ is a subgroup of $\mathcal{L}$, and $N_{\mathcal{F}}(R) = \mathcal{F}_{N_S(R)}(N_{\mathcal{L}}(R))$.

For each $V \in R^{\mathcal{F}}$ let $\mathbf{Y}_V$ be the set of elements $y \in \mathcal{L}$ such that $R^y = V$ and such that $N_S(V) \leq S_{y^{-1}}$. Define $\mathbf{Y}$ to be the union of the sets $\mathbf{Y}_V$, taken over all $V \in R^{\mathcal{F}}$. Notice that points (b) and (c) of 3.1 are equivalent to the condition that $\mathbf{Y}_V$ be non-empty for each $V \in R^{\mathcal{F}}$.

The remainder of this section will be devoted to proving the following result (in which the notion of restriction of a locality is given by 2.11).

Theorem 3.3. *Assume Hypothesis 3.2, and set $\Delta^+ = \Delta \bigcup R^{\mathcal{F}}$. Then there exists a proper locality $(\mathcal{L}^+, \Delta^+, S)$ such that $\mathcal{L}$ is the restriction of $\mathcal{L}^+$ to Δ, and such that $N_{\mathcal{L}^+}(R) = N_{\mathcal{L}}(R)$. Moreover, we then have $\mathcal{F}_S(\mathcal{L}^+) = \mathcal{F}$, and the following hold.*

(a) *For any locality $(\widetilde{\mathcal{L}}, \Delta^+, S)$ such that $\widetilde{\mathcal{L}} \mid_{\Delta} = \mathcal{L}$, and such that $N_{\widetilde{\mathcal{L}}}(R) = N_{\mathcal{L}}(R)$, we have $\mathcal{F}_S(\widetilde{\mathcal{L}}) = \mathcal{F}$, and there is a unique isomorphism $\beta : \mathcal{L}^+ \to \widetilde{\mathcal{L}}$ which restricts to the identity map on $\mathcal{L}$.*

(b) *Let $\mathcal{L}_0^+$ be the set of all $g \in \mathcal{L}^+$ such that S_g contains an $\mathcal{F}$-conjugate of R. Then $\mathcal{L}_0^+$ is a partial subgroup of $\mathcal{L}$, and $\mathcal{L}^+$ is the pushout in the category of partial groups of the diagram*

$$\mathcal{L}_0^+ \leftarrow \mathcal{L} \cap \mathcal{L}_0 \rightarrow \mathcal{L}$$

of inclusion homomorphisms.

Along the way to proving Theorem 3.3, we shall explicitly determine the partial group structure of $\mathcal{L}^+$ in 3.9 through 3.13. These results will then play an important computational role in section 4 and in section 7. —

Set $\mathbf{X} = \{x^{-1} \mid x \in \mathbf{Y}\}$ and set

$$\Phi = \mathbf{X} \times N_{\mathcal{L}}(R) \times \mathbf{Y}.$$

For any $\phi = (x^{-1}, h, y) \in \Phi$ set $U_\phi = R^x$ and $V_\phi = R^y$. Thus:

$$U_\phi \xrightarrow{x^{-1}} R \xrightarrow{h} R \xrightarrow{y} V_\phi, \quad \text{and} \quad N_S(U_\phi) \xrightarrow{x^{-1}} N_S(R) \xleftarrow{y^{-1}} N_S(V_\phi)$$

are diagrams of conjugation maps, labelled by the conjugating elements. (In the first of these diagrams the conjugation maps are isomorphisms, and in the second they are homomorphisms into $N_S(R)$.)

(3.4). Define a relation $\sim$ on Φ as follows. For $\phi = (x^{-1}, h, y)$ and $\overline{\phi} = (\overline{x}^{-1}, \overline{h}, \overline{y})$ in Φ, write $\phi \sim \overline{\phi}$ if

(i) $U_\phi = U_{\overline{\phi}}$, $V_\phi = V_{\overline{\phi}}$, and
(ii) $(\overline{x}x^{-1})h = \overline{h}(\overline{y}y^{-1})$.

The products in 3.4(ii) are well-defined. Namely, by 3.4(i), $(\overline{x}, x^{-1}) \in \mathbf{D}$ via $N_S(U)^{\overline{x}^{-1}}$, $\overline{x}x^{-1} \in N_{\mathcal{L}}(R)$, and then $(\overline{x}x^{-1}, h) \in \mathbf{D}$ since $N_{\mathcal{L}}(R)$ is a group (3.2(3)). The same considerations apply to $(\overline{y}, y)$ and $(\overline{h}, \overline{y}y^{-1})$.

One may depict the relation $\sim$ by means of a commutative diagram, as follows.

$$\begin{tikzcd}
U \arrow[r, "x^{-1}"] \arrow[d, equal] & R \arrow[r, "g"] & R \arrow[r, "y"] & V \arrow[d, equal] \\
U \arrow[r, "\overline{x}^{-1}"'] & R \arrow[r, "\overline{g}"'] \arrow[u, "\overline{x}x^{-1}"] & R \arrow[r, "\overline{y}"'] \arrow[u, "\overline{y}y^{-1}"'] & V
\end{tikzcd}$$

Lemma 3.5. $\sim$ *is an equivalence relation on* Φ.

Proof. Evidently $\sim$ is reflexive and symmetric. Let $\phi_i = (x_i^{-1}, g_i, y_i) \in \Phi$ $(1 \leq i \leq 3)$ with $\phi_1 \sim \phi_2 \sim \phi_3$. Then $R^{x_1} = R^{x_3}$ and $R^{y_1} = R^{y_3}$. Notice that

$$(x_3, x_2^{-1}, x_2, x_1^{-1}) \in \mathbf{D} \quad \text{via } N_S(U)^{x_3^{-1}} \text{ and,}$$
$$(y_3, y_2^{-1}, y_2, y_1^{-1}) \in \mathbf{D} \quad \text{via } N_S(V)^{y_3^{-1}}.$$

Computation in the group $N_{\mathcal{L}}(R)$ then yields

$$
\begin{aligned}
(x_3x_1^{-1})g_1 &= (x_3x_2^{-1})(x_2x_1^{-1})g_1 = (x_3x_2^{-1})g_2(y_2y_1^{-1}) \\
&= g_3(y_3y_2^{-1})(y_2y_1^{-1}) = g_3(y_3y_1^{-1}),
\end{aligned}
$$

which completes the proof of transitivity. $\square$

Lemma 3.6. *Let $\psi \in \Phi$ and set $U = U_\psi$ and $V = V_\phi$. Let $x \in \mathbf{Y}_U$ and $y \in \mathbf{Y}_V$. Then there exists a unique $h \in N_{\mathcal{L}}(V)$ such that $\psi \sim (x^{-1}, h, y)$.*

Proof. Write $\psi = (\overline{x}^{-1}, \overline{h}, \overline{y})$. Then $(\overline{x}, x^{-1}) \in \mathbf{D}$ via $N_S(U)^{\overline{x}^{-1}}$, and $\overline{x}x^{-1} \in N_{\mathcal{L}}(R)$. Similarly $(\overline{y}, y^{-1}) \in \mathbf{D}$ and $\overline{y}y^{-1} \in N_{\mathcal{L}}(R)$. As $N_{\mathcal{L}}(R)$ is a subgroup of $\mathcal{L}$ we may form the product $h := (x\overline{x}^{-1})\overline{h}(\overline{y}y^{-1})$ and obtain $(\overline{x}x^{-1})h = \overline{h}(\overline{y}y^{-1})$. Setting $\phi = (x^{-1}, h, y)$, we thus have $\phi \sim \psi$. If $h' \in N_{\mathcal{L}}(R)$ with also $(x^{-1}, h', y) \sim \psi$ then $h' = (x\overline{x}^{-1})\overline{h}(\overline{y}y^{-1}) = h$. $\square$

For ease of reference we record the following observation, even though it is simply part of the definition of the relation $\sim$.

Lemma 3.7. *Let C be a $\sim$-class of Φ, let $\phi = (x^{-1}, g, y) \in C$, and set $U = R^x$ and $V = R^y$. Then the pair (U, V) depends only on C, and not on the choice of representative ϕ.* $\square$

Lemma 3.8. *Let $\mathcal{L}_0$ be the set of all $g \in \mathcal{L}$ such that S_g contains an $\mathcal{F}$-conjugate U of R. For each $g \in \mathcal{L}_0$ set*

$$
\Phi_g = \{\phi \in \Phi \cap \mathbf{D} \mid \Pi(\phi) = g\},
$$

and set

$$
\mathcal{U}_g = \{U \in R^{\mathcal{F}} \mid U \leq S_g\}.
$$

(a) *$\mathcal{U}_g$ is the set of all U_ϕ such that $\phi \in \Phi_g$.*
(b) *Φ_g is a union of $\sim$-classes.*
(c) *Let $U \in \mathcal{U}_g$, set $V = U^g$, and let $x \in \mathbf{X}_U$ and $y \in \mathbf{X}_V$. Then $(x, g, y^{-1}) \in \mathbf{D}$, $h := xgy^{-1} \in N_{\mathcal{L}}(U)$, and $\phi := (x^{-1}, h, y) \in \Phi_g$. If also $(x^{-1}, h', y) \in \Phi_g$ then $h = h'$.*
(d) *Let ϕ and ψ in Φ_g. Then*

$$
\phi \sim \psi \iff U_\phi = U_\psi \iff V_\phi = V_\psi.
$$

Proof. Let $g \in \mathcal{L}_0$, let $U \in \mathcal{U}_g$, and set $V = U^g$. Let $(x^{-1}, y) \in \mathbf{X}_U \times \mathbf{Y}_V$ and set $w = (x, g, y^{-1})$. Then $w \in \mathbf{D}$ via

$$
(N_{S_g}(U)^{x^{-1}}, N_{S_g}(U), N_{S_{g^{-1}}}(V), N_{S_{g^{-1}}}(V)^{y^{-1}}).
$$

Set $h = \Pi(w)$. Then $h \in N_{\mathcal{L}}(R)$ since $R^x = U$ and $R^y = V$. Set $w' = (x^{-1}, x, g, y^{-1}, y)$. Then $w' \in \mathbf{D}$ via $N_{S_w}(U)$, and $g = \Pi(w') = \Pi(x^{-1}, h, y)$ by $\mathbf{D}$-associativity (I.1.4(b)).

If also $(x^{-1}, h', y) \in \mathbf{D}$ with $\Pi(x^{-1}, h', y) = g$ then $h = h'$ by the cancellation rule (I.1.4(e)). This establishes (c), and shows that $\mathcal{U}_g \subseteq \{U_\phi \mid \phi \in \Phi_g\}$. The opposite inclusion is immediate (cf. I.2.3(c)), so also (a) is established.

Let $\phi \in \Phi_g$ and let $\overline{\phi} \in \Phi$ with $\phi \sim \overline{\phi}$. Write $\phi = (x^{-1}, h, y)$ and $\overline{\phi} = (\overline{x}^{-1}, \overline{h}, \overline{y})$, and set

$$(*) \qquad w' = (\overline{x}^{-1}, \overline{x}, x^{-1}, h, y, \overline{y}^{-1}, \overline{y}).$$

As $\phi \in \mathbf{D}$ we have $N_{S_\phi}(U) \in \Delta$ by 3.2(1), and then $w' \in \mathbf{D}$ via $N_{S_\phi}(U)$. Now $\Pi(w') = \Pi(\phi) = g$, while also

$$(**) \qquad \Pi(w') = \Pi(\overline{x}^{-1}, \Pi((\overline{x}x^{-1}), h, (y\overline{y}^{-1})), \overline{y}) = \Pi(\overline{x}^{-1}, \overline{h}, \overline{y}) = \Pi(\overline{\phi}),$$

and thus $\overline{\phi} \in \Phi_g$. This proves (b).

It remains to prove (d). So, let $\phi, \psi \in \Phi_g$. If $\phi \sim \psi$ then $U_\phi = U_\psi$ and $V_\phi = V_\psi$ by 3.7. On the other hand, assume that $U_\phi = U_\psi$ or that $V_\phi = V_\psi$. Then both equalities obtain, since $V_\phi = (U_\phi)^g$ and $V_\psi = (U_\psi)^g$. Write $\phi = (x^{-1}, h, y)$ and $\psi = (\overline{x}^{-1}, \overline{h}, \overline{y})$ in the usual way, and define w' as in (*). Then $w' \in \mathbf{D}$ via $N_{S_\phi}(V_\phi)$, and $\Pi(w') = \Pi(\phi) = g$. Set

$$h' = \Pi((\overline{x}x^{-1})h(y\overline{y}^{-1})).$$

Then $g = \Pi(w') = \Pi(\overline{x}^{-1}, h', \overline{y})$, so $(\overline{x}^{-1}, h', \overline{y}) \in \Phi_g$. Then 3.6 yields $h' = \overline{h}$, and thus $(\overline{x}^{-1}, h', \overline{y}) = \overline{\phi}$. This shows that $\phi \sim \overline{\phi}$, completing the proof of (d). □

(3.9). We now have a partition of the disjoint union $\mathcal{L} \bigsqcup \Phi$ (and a corresponding equivalence relation $\approx$ on $\mathcal{L} \bigsqcup \Phi$) by means of three types of $\approx$-classes, as follows.

- Singletons $\{f\}$, where $f \in \mathcal{L}$ and where S_f contains no $\mathcal{F}$-conjugate of R (classes whose intersection with Φ is empty).
- $\sim$-classes $[\phi]$ such that $[\phi] \cap \mathbf{D} = \emptyset$ (classes whose intersection with $\mathcal{L}$ is empty).
- Classes $\Phi_g \cup \{g\}$, where $g \in \mathcal{L}$ and where S_g contains an $\mathcal{F}$-conjugate of R (classes having a non-empty intersection with both $\mathcal{L}$ and Φ).

For any element $E \in \mathcal{L} \cup \Phi$, write $[E]$ for the $\approx$-class of E. (Thus $[E]$ is also a $\sim$-class if and only if $[E] \cap \mathbf{D} = \emptyset$.) Let $\mathcal{L}^+$ be the set of all $\approx$-classes. Let $\mathcal{L}_0^+$ be the subset of $\mathcal{L}^+$, consisting of those $\approx$-classes whose intersection with Φ is non-empty. That is, the members of $\mathcal{L}_0^+$ are the $\approx$-classes of the form $[\phi]$ for some $\phi \in \Phi$.

Recall from Part I that there is an inversion map $w \mapsto w^{-1}$ on $\mathbf{W}(\mathcal{L})$, given by $(g_1, \cdots, g_n)^{-1} = (g_n^{-1}, \cdots, g_1^{-1})$. The following result is then a straightforward consequence of the definitions of Φ, $\sim$, and $\approx$.

Lemma 3.10. *The inversion map on $\mathbf{W}(\mathcal{L})$ preserves Φ. Further, for each $E \in \mathcal{L} \cup \Phi$ we have $E^{-1} \in \mathcal{L} \bigsqcup \Phi$, and $[E^{-1}]$ is the set $[E]^{-1}$ of inverses of members of $[E]$.* □

Definition 3.11. For any $\gamma = (\phi_1, \cdots, \phi_n) \in \mathbf{W}(\Phi)$ let w_γ be the word $\phi_1 \circ \cdots \circ \phi_n$ in $\mathbf{W}(\mathcal{L})$. Let Γ be the set of all $\gamma \in \mathbf{W}(\Phi)$ such that S_{w_γ} contains an $\mathcal{F}$-conjugate of R. Let $\mathbf{D}_0^+$ be the set of all sequences $w = ([\phi_1], \cdots, [\phi_n]) \in \mathbf{W}(\mathcal{L}_0^+)$ for which there exists a sequence γ of representatives for w with $\gamma \in \Gamma$. We shall say that γ is a Γ-*form* of w.

The following lemma shows how to define a product $\Pi_0^+ : \mathbf{D}_0^+ \to \mathcal{L}_0^+$.

Lemma 3.12. *Let $w = ([\phi_1], \cdots, [\phi_n]) \in \mathbf{D}_0^+$, and let $\gamma = (\phi_1, \cdots, \phi_n)$ be a Γ-form of w. Write $\phi_i = (x_i^{-1}, h_i, y_i)$.*

(a) *$(y_i, x_{i+1}^{-1}) \in \mathbf{D}$ and $y_i x_{i+1}^{-1} \in N_{\mathcal{L}}(R)$ for each i with $1 \le i < n$.*

(b) *Set*

$$w_0 = (h_1, y_1 x_2^{-1}, \cdots, y_{n-1} x_n^{-1}, h_n).$$

Then $w_0 \in \mathbf{W}(N_{\mathcal{L}}(R))$ and $(x_1^{-1}, \Pi(w_0), y_n) \in \Phi$. Moreover:

(c) *The $\approx$-class $[x_1^{-1}, \Pi(w_0), y_n]$ depends only on w, and not on the choice of Γ-form of w.*

Proof. Let $U \in R^{\mathcal{F}}$ and let $x, y \in \mathbf{Y}_U$. Then $(y, x^{-1}) \in \mathbf{D}$ via $N_S(U)^{y^{-1}}$, and then $yx^{-1} \in N_{\mathcal{L}}(R)$. This proves (a), and shows that $w_0 \in \mathbf{W}(N_{\mathcal{L}}(R))$. As $N_{\mathcal{L}}(R))$ is a subgroup of $\mathcal{L}$, (b) follows.

Let $\overline{\gamma} = (\overline{\phi}_1, \cdots, \overline{\phi}_n)$ be any Γ-form of w, write $\overline{\phi}_i = (\overline{x}_i^{-1}, \overline{h}_i, \overline{y}_i)$, and define $\overline{w}_0$ in analogy with w_0. Set $U_0 = U_{\phi_1}$ and $\overline{U}_0 = U_{\overline{\phi}_1}$. For each i with $1 \le i \le n$ set $U_i = V_{\phi_i}$ and $\overline{V}_i = U_{\overline{\phi}_i}$. Thus:

$$U_{i-1} \xrightarrow{x_i^{-1}} R \xrightarrow{h_i} R \xrightarrow{y_i} U_i \quad \text{and} \quad \overline{U}_{i-1} \xrightarrow{\overline{x}_i^{-1}} R \xrightarrow{\overline{h}_i} R \xrightarrow{\overline{y}_i} \overline{U}_i.$$

Suppose that there exists an index j with $U_j \ne \overline{U}_j$. As $\phi_j \approx \overline{\phi}_j$ it follows from 3.5 and 3.6that ϕ_j and $\overline{\phi}_j$ are in $\mathbf{D}$, and that there is an element $g_j \in \mathcal{L}_0$ such that ϕ_j and $\overline{\phi}_j$ are in Φ_g. If $j < n$ then

$$U_{j+1} = (U_j)^g \ne (\overline{U}_j)^g = \overline{U}_{j+1},$$

and if $j > 0$ one obtains $U_{j-1} \ne \overline{U}_{j-1}$ in similar fashion, by consideration of w^{-1} via 3.9. Thus, for each index i we have $U_i \ne \overline{U}_i$. Then ϕ_i and $\overline{\phi}$ lie in distinct $\sim$-classes by 3.5. As $\phi_i \approx \overline{\phi}_i$ it follows that ϕ_i and $\overline{\phi}_i$ are members of $\Phi \cap \mathbf{D}$, and that there is a word $v = (g_1, \cdots, g_n) \in \mathbf{W}(\mathcal{L})$ with $g_i = \Pi(\phi_i) = \Pi(\overline{\phi}_i)$. As $\langle U_0, \overline{U}_0 \rangle \le S_v$, we have $v \in \mathbf{D}$.

Set $P = S_v$ and set $P_0 = N_P(U_0)$. Then $P_0 \in \Delta$ by 3.2(1). Set $P_i = (P_{i-1})^{g_i}$ for $1 \le i \le n$. Then

$$P_{i-1}^{x_i^{-1}} \le N_S(R) \ \text{ and } \ (P_i)^{y_i^{-1}} \le N_S(R).$$

Thus conjugation by g_i maps $P_{i-1}^{x_i^{-1}}$ to $(P_i)^{y_i^{-1}}$, and this shows:

(*) Set $w_\gamma = \gamma_1 \circ \cdots \circ \gamma_n$. Then $w_\gamma \in \mathbf{D}$ via P_0.

Similarly, one obtains $w_{\overline{\gamma}} \in \mathbf{D}$ via $N_P(\overline{U}_0)$, where Now $\mathbf{D}$-associativity yields:

$$\Pi(x_1^{-1}, \Pi(w_0), y_n) = \Pi(w_\gamma) = \Pi(v) = \Pi(w_{\overline{\gamma}}) = \Pi(\overline{x}_1^{-1}, \Pi(\overline{w}_0), \overline{y}_n).$$

Thus $\Pi(x_1^{-1}, \Pi(w_0), y_n)$ and $(\overline{x}_1^{-1}, \Pi(\overline{w}_0), \overline{y}_n)$ lie in the same fiber of $\Pi : \Phi \cap \mathbf{D} \to \mathcal{L}_0$, and thus $(x_1^{-1}, \Pi(w_0), y_n) \approx (\overline{x}_1^{-1}, \Pi(\overline{w}_0), \overline{y}_n)$. This reduces (c) to the following claim.

(**) Suppose that $U_i = \overline{U}_i$ for all i. Then $(x_1^{-1}, \Pi(w_0), y_n) \sim (\overline{x}_1^{-1}, \Pi(\overline{w}_0), \overline{y}_n)$.

Among all counter-examples to (**), let w be chosen so that n is as small as possible. Then $n \neq 0$ (i.e. w and $\overline{w}$ are non-empty words), as there is otherwise nothing to verify. If $n = 1$ (so that $w = (\phi_1)$ and $\overline{w} = (\overline{\phi}_1)$) then $w_0 = (h)$, $\overline{w}_0 = (\overline{h})$, and (**) follows since $\phi \sim \overline{\phi}$. Thus, $n \geq 2$.

Set $\psi = (x_1^{-1}, h_1(y_1 x_2^{-1})h_2, y_2)$, and similarly define $\overline{\psi}$. Then ψ and $\overline{\psi}$ are in Φ. As $\phi_i \sim \overline{\phi}_i$ for $i = 1, 2$, one has the commutative diagram

$$\begin{tikzcd}
U_0 \arrow[r, "x_1^{-1}"] \arrow[d, equal] & R \arrow[r, "h_1"] & R \arrow[r, "y_1"] & U_1 \arrow[r, "x_2^{-1}"] \arrow[d, equal, "A"'] & R \arrow[r, "h_2"] & R \arrow[r, "y_2"] & U_2 \arrow[d, equal] \\
U_0 \arrow[r, "\overline{x}_1^{-1}"] & R \arrow[r, "\overline{h}_1"] \arrow[u, "\overline{x}_1 x_1^{-1}"'] & R \arrow[r, "\overline{y}_1"] \arrow[u, "\overline{y}_1 y_1^{-1}"] & U_1 \arrow[r, "\overline{x}_2^{-1}"] & R \arrow[r, "\overline{h}_2"] \arrow[u, "\overline{x}_2 x_2^{-1}"'] & R \arrow[r, "\overline{y}_2"] \arrow[u, "\overline{y}_2 y_2^{-1}"] & U_2
\end{tikzcd}$$

of conjugation maps. This diagram then collapses to the commutative diagram

$$\begin{tikzcd}
U_0 \arrow[r, "x_1^{-1}"] \arrow[d, equal] & R \arrow[r, "h_1(y_1 x_2^{-1})h_2"] & R \arrow[r, "y_2"] & U_2 \arrow[d, equal] \\
U_0 \arrow[r, "\overline{x}_1^{-1}"] & R \arrow[r, "\overline{h}_1(\overline{y}_1 \overline{x}_2^{-1})\overline{h}_2"] \arrow[u, "\overline{x}_1 x_1^{-1}"'] & R \arrow[r, "\overline{y}_2"] \arrow[u, "\overline{y}_2 y_2^{-1}"] & U_2
\end{tikzcd}$$

which shows that $\psi \sim \overline{\psi}$. Here $(\psi, \phi_3, \cdots, \phi_n)$ and $(\overline{\psi}, \overline{\phi}_3, \cdots, \overline{\phi}_n)$ are Γ-forms of the word $u = ([\psi], [\phi_3], \cdots, [\phi_n])$. Set

$$u_0 = (h_1(y_1 x_2^{-1})h_2, y_2 x_3^{-1}, \cdots, y_{n-1} x_n^{-1}, h_n),$$

and similarly define $\overline{u}_0$. Then $\Pi(u_0) = \Pi(w_0)$ and $\Pi(\overline{u}_0) = \Pi(\overline{w}_0)$. The minimality of n then yields

$$(x^{-1}, \Pi(w_0), y_n) \approx (\overline{x}_1^{-1}, \Pi(\overline{w}_0), \overline{y}_n).$$

This proves (**), and thereby completes the proof of (c). $\square$

Proposition 3.13. *There is a mapping $\Pi^+ : \mathbf{D}_0^+ \to \mathcal{L}_0^+$, given by*

$$\Pi_0^+(\emptyset) = [\mathbf{1}, \mathbf{1}, \mathbf{1}],$$

and by

(*) $$\Pi_0^+(w) = [x_1^{-1}, \Pi(w_0), y_n],$$

on non-empty words $w \in \mathbf{D}_0^+$; where w_0 is given by a Γ-form of w as in 3.10(b). Further, there is an involutory bijection on $\mathcal{L}_0^+$ given by

$$(**) \qquad [x^{-1}, g, y]^{-1} = [y^{-1}, g^{-1}, x].$$

With these structures, $\mathcal{L}_0^+$ is a partial group.

Proof. The reader may refer to I.1.1 for the conditions (1) through (4) defining the notion of partial group. Condition (1) requires that $\mathbf{D}_0^+$ contain all words of length 1 in the alphabet $\mathcal{L}_0^+$, and that $\mathbf{D}_0^+$ be closed with respect to decomposition. (That is, if u and v are two words in the free monoid $\mathbf{W}(\mathcal{L}_0^+)$, and the concatenation $u \circ v$ is in $\mathbf{D}_0^+$, then u and v are in $\mathbf{D}_0^+$.) Both of these conditions are immediate consequences of the definition of $\mathbf{D}_0^+$.

That Π_0^+ is a well-defined mapping is given by 3.10. The proof that Π_0^+ satisfies the conditions I.1.1(2) (Π_0^+ restricts to the identity map on words of length 1) and I.1.1(3):

$$\mathbf{u} \circ \mathbf{v} \circ \widetilde{\in} \mathbf{D}_0^+ \implies \Pi_0^+(\mathbf{u} \circ \mathbf{v} \circ \mathbf{v}) = \Pi_0^+(\mathbf{u} \circ \Pi_0^+(\mathbf{v}) \circ \mathbf{w})$$

are then straightforward, and may safely be omitted.

The inversion map $[x^{-1}, h, y] \mapsto [y^{-1}, h^{-1}, x]$ is well-defined by 3.8. Evidently this mapping is an involutory bijection, and it extends to an involutory bijection

$$(C_1, \cdots, C_n)^{-1} = (C_n^{-1}, \cdots, C_1^{-1})$$

on $\mathbf{W}(\mathcal{L}_0^+)$. It thus remains to show I.1.1(4). That is, we must check that

$$w \in \mathbf{D}_0^+ \implies w^{-1} \circ w \in \mathbf{D}_0^+ \text{ and } \Pi_0^+(w^{-1} \circ w) = [\mathbf{1}, \mathbf{1}, \mathbf{1}].$$

In detail: take $w = (C_1, \cdots, C_n)$ and let $\gamma = (\phi_1, \cdots, \phi_n)$ be a Γ-form of w, where ϕ_i is written as $[x_i^{-1}, h_i, y_i)$. One easily verifies that $\gamma^{-1} \circ \gamma \in \Gamma$, and hence $w^{-1} \circ w \in \mathbf{D}_0^+$. Now

$$\Pi_0 + (w^{-1} \circ w) = [y_n^{-1}, \Pi(u_0), y_n],$$

where

$$u_0 = (g_n^{-1}, x_n y_{n-1}^{-1}, \cdots, x_2 y_1^{-1}, g_1^{-1}, x_1 x_1^{-1}, g_1, y_1 x_2^{-1}), \cdots, y_{n-1} x_n^{-1}, g_n).$$

One observes that $\Pi(u_0) = \mathbf{1}$, and so $\Pi_0 + (w^{-1} \circ w) = [y_n^{-1}, \mathbf{1}, y_n]$. Now observe that $(y_n^{-1}, \mathbf{1}, y_n) \equiv (\mathbf{1}, \mathbf{1}, \mathbf{1})$, since $(y_n^{-1}, \mathbf{1}, y_n)$ and $(\mathbf{1}, \mathbf{1}, \mathbf{1})$ are in $\mathbf{D}$ and since $\Pi(y_n^{-1}, \mathbf{1}, y_n) = \mathbf{1} = \Pi(\mathbf{1}, \mathbf{1}, \mathbf{1})$. Thus $\Pi_0 + (w^{-1} \circ w) = [\mathbf{1}, \mathbf{1}, \mathbf{1}] = \Pi_0^+(\emptyset)$. Thus I.1.1(4) holds, and the proof is complete. $\square$

Lemma 3.14. *Let $\mathbf{D}_0$ be the set of all $w \in \mathbf{D}$ such that S_w contains an $\mathcal{F}$-conjugate of R, and let $\mathcal{L}_0$ be the set of words of length 1 in $\mathbf{D}_0$, regarded as a subset of $\mathcal{L}$. Let $\Pi_0 : \mathbf{D}_0 \to \mathcal{L}_0$ be the restriction of Π to $\mathbf{D}_0$. Then:*

(a) *$\mathcal{L}_0$, with Π_0 and the restriction of the inversion map on $\mathcal{L}$ to $\mathcal{L}_0$, is a partial group.*

(b) *Let $\iota_0 : \mathcal{L}_0 \to \mathcal{L}$ be the inclusion map, and let $\lambda_0 : \mathcal{L}_0 \to \mathcal{L}_0^+$ be the mapping $g \mapsto \Phi_g \cup \{g\}$. Then ι_0 and λ_0 are homomorphisms of partial groups.*

Proof. The verification of (a) is straightforward, and is left to the reader (see I.1.1). Moreover, since the product in $\mathcal{L}_0$ is inherited from $\mathcal{L}$, it is immediate that ι_0 is a homomorphism of partial groups.

Let $v = (g_1, \cdots, g_n) \in \mathbf{D}_0$, set $w = ([g_1], \cdots, [g_n])$, and let $U \in R^{\mathcal{F}}$ be chosen so that $U \leq S_v$. By 3.12 there exists a word $\gamma = (\phi_1, \cdots, \phi_n) \in \mathbf{W}(\Phi)$ such that $\phi_i \in [g_i]$ and such that $U \leq S_{w_\gamma}$, where w_γ is the word $\phi_1 \circ \cdots \circ \phi_n \in \mathbf{W}(\mathcal{L})$. Then γ is a Γ-form of w, and so $w \in \mathbf{D}_0^+$. Set $P = S_w$. The proof of the intermediary result (*) in the proof of 3.10 shows that $w_\gamma \in \mathbf{D}$ via $N_P(U)$. Write $\phi_i = (x_i^{-1}, h_i, y_i)$, and form the word $w_0 \in \mathbf{W}(N_{\mathcal{L}}(R))$ as in 3.10(b). Set $g = \Pi(v)$. Then $g = \Pi(w_\gamma) = \Pi(x_1^{-1}, w_0, y_n)$ by $\mathbf{D}$-associativity, and thus $(x_1^{-1}, w_0, y_n) \in \Phi_g$. That is, we have $[g] = [x_1^{-1}, w_0, y_n]$. This shows that λ_0 is a homomorphism of partial groups, completing the proof of (b). $\square$

We remark that it is easily verified that $Im(\lambda_0)$ is in fact a partial subgroup of $\mathcal{L}_0^+$. But there is no reason to suppose that $Im(\iota_0)$ is a partial subgroup of $\mathcal{L}$, as it may be the case that $\mathbf{W}(\mathcal{L}_0) \cap \mathbf{D}$ is not contained in $\mathbf{D}_0$.

For any $w = (g_1, \cdots, g_n) \in \mathbf{W}(\mathcal{L})$ set $[w] = ([g_1], \cdots, [g_n])$. For any subset W of $\mathbf{W}(\mathcal{L})$ set $[X] = \{[w] \mid w \in W\}$. Since each $\approx$-class $[g_i]$ intersects $\mathcal{L}$ in $\{g_i\}$ it follows that the product $\Pi : \mathbf{D} \to \mathcal{L}$ may be regarded as a mapping $[\mathbf{D}] \to \mathcal{L}$.

Recall from Theorem I.1.17 that the category of partial groups has all colimits (and all limits). In particular, pushouts are available.

Proposition 3.15. *Set $\mathbf{D}^+ = [\mathbf{D}] \cup \mathbf{D}_0^+$.*

(a) *The products $\Pi : \mathbf{D} \to \mathcal{L}$ and $\Pi_0^+ : \mathbf{D}_0^+ \to \mathcal{L}_0^+$ agree on $[\mathbf{D}] \cap \mathbf{D}_0^+$, and $\Pi \cup \Pi_0^+$ is a mapping*
$$\Pi^+ : \mathbf{D}^+ \to \mathcal{L}^+.$$

(b) *$\mathcal{L}^+$ is a partial group via the product Π^+ and the involutory bijection given by 3.8.*

(c) *Let $\lambda : \mathcal{L} \to \mathcal{L}^+$ be the mapping $g \mapsto [g]$, and let $\iota : \mathcal{L}_0^+ \to \mathcal{L}^+$ be the inclusion map. Then λ and ι are injective homomorphisms of partial groups, and*

$$\begin{array}{ccc} \mathcal{L}_0^+ & \xrightarrow{\iota} & \mathcal{L}^+ \\ {\scriptstyle\lambda_0}\uparrow & & \uparrow{\scriptstyle\lambda} \\ \mathcal{L}_0 & \xrightarrow[\iota_0]{} & \mathcal{L} \end{array}$$

is a pushout diagram in the category of partial groups.

Proof. That Π and Π_0^+ agree on $[\mathbf{D}] \cap \mathbf{D}_0^+$ is one way of interpreting the fact (3.12(b)) that λ_0 and ι_0 are homomorphisms of partial groups. Thus (a) holds, and point (b) is then a straightforward exercise with definition I.1.1.

Let $\mathcal{L}^*$ (with the appropriate diagram of homomorphisms) be a pushout for

$$(*) \qquad \mathcal{L}_0^+ \xleftarrow{\lambda_0} \mathcal{L}_0 \xrightarrow{\iota_0} \mathcal{L}.$$

By I.1.17 we may in fact take the underlying set of $\mathcal{L}^*$ to be the standard pushout of (*) as a diagram of mappings of sets. That is, we may take $\mathcal{L}^*$ to be the disjoint union $\mathcal{L}_0^+ \bigsqcup \mathcal{L}$ modulo the relation $\equiv$ which identifies $g \in \mathcal{L}_0$ with $g\lambda_0$. Here $g\lambda_0 = \Phi_g \cup \{g\}$, and the elements of $\mathcal{L}$ which are not in $\mathcal{L}_0$ are by definition the singletons $\{f\}$ such that S_f contains no $\mathcal{F}$-conjugate of R. By identifying such a singleton $\{f\}$ with its unique element we thereby obtain $\mathcal{L}^* = \mathcal{L}^+$ as sets.

The domain $\mathbf{D}^*$ of the product in $\mathcal{L}^*$ is obtained by from the disjoint union $\mathbf{D}_0^+ \bigsqcup \mathbf{D}$ by imposing the $\equiv$-relation componentwise. That is, we have $\mathbf{D}^* = \mathbf{D}^+$. The product $\Pi^* : \mathbf{D}^* \to \mathcal{L}^*$ is then the union of the products Π_0^* and Π; which is to say that $\Pi^* = \Pi^+$. Similarly, the inversion maps on $\mathcal{L}^*$ and on $\mathcal{L}^+$ coincide, and so $\mathcal{L}^* = \mathcal{L}^+$ as partial groups. That λ and ι are the homomorphisms which give the required pushout diagram in then immediate.

Now let $f, g \in \mathcal{L}$ with $f\lambda = g\lambda$. Then 3.7 yields

$$\{f\} = [f] \cap \mathcal{L} = [g] \cap \mathcal{L} = \{g\}$$

and so λ is injective. The inclusion map ι is of course injective, so the proof is complete. $\square$

Let Δ^+ be the union of Δ with the set of all subgroups $P \leq S$ such that P contains an $\mathcal{F}$-conjugate of R. The following lemma prepares the way for showing that $(\mathcal{L}^+, \Delta^+)$ is an objective partial group.

Lemma 3.16. *Let $\lambda : \mathcal{L} \to \mathcal{L}^+$ be the homomorphism of partial groups given by $g \mapsto [g]$. Let $[\phi] \in \mathcal{L}_0^+$, and let $S_{[\phi]}$ be the set of all $a \in S$ such that $[a]^{[\phi]}$ is defined in $\mathcal{L}^+$, and such that $[a]^{[\phi]} \in [S]$. Then $S_{[\phi]} = S_\phi$.*

Proof. Let $a \in S$ such that $[a]^{[\phi]} = [b]$ for some $b \in S$. Suppose first that $[\phi] \cap \mathcal{L}$ is non-empty, and let g be the unique element of $[\phi] \cap \mathcal{L}$. The equality $[a]^{[\phi]} = [b]$ then simply means that $(a^g)\lambda = b\lambda$, and the injectivity of λ (3.15(c)) yields $a^g = b$. Thus $a \in S_g$. Conversely, for any $x \in S_g$ we have $x\lambda = [x] \in S_{[\phi]} = g\lambda$, and thus the lemma holds in this case.

Assume now that $[\phi] \cap \mathcal{L} = \emptyset$. Then $[\phi]$ is a $\sim$-class by 3.9, and 3.7 shows that the pair $(U, V) := (U_\phi, V_\phi)$ is constant over all $\phi \in [\phi]$. Set $w = ([\phi]^{-1}, [a], [\phi])$. Then $w \in \mathbf{D}_0^+$ by hypothesis, so there is a Γ-form $\gamma = (\phi^{-1}, \psi, \overline{\phi})$ of w. This means that, upon setting $w\gamma = \phi^{-1} \circ \psi \circ \overline{\phi}$, we have $V \leq S_{w_\gamma}$. The uniqueness of (U, V) for $[\phi]$ (and of (V, U) for $[\phi]^{-1}$) yields $U = U_\psi = V_\psi$, and then $a \in N_S(U)$ since $a = \Pi(\psi)$. Since $[b]^{[\phi]^{-1}} = [a]$ we

similarly obtain $b \in N_S(V)$. Notice also that since (U, V) is independant of the choice of representative for $[\phi]$ we may take $\overline{\phi}$ to be ϕ. As $x \in \mathbf{Y}_U$ and $y \in \mathbf{Y}_V$ we obtain:

(*) $a^{x^{-1}}, b^{y^{-1}} \in N_S(R)$.

Write $\phi = (x^{-1}, h, y)$, and set $\theta = (a^{-1}x^{-1}, \mathbf{1}, xa^2)$. As $a \in N_S(U)$ we get $xa^2 \in \mathbf{Y}_U$ and $a^{-1}x^{-1} \in \mathbf{X}_U$. Thus $\theta \in \Phi$. Moreover, we have $\theta \in \mathbf{D}$ via $N_S(U)$, and $\Pi(\theta) = a$. Thus $\theta \in \Phi_a$, and $(\phi^{-1}, \theta, \phi)$ is a Γ-form of w. Then

$$\Pi^+(w) = [y^{-1}, h^{-1}(x(a^{-1}x^{-1}))((xa^2)x^{-1})h, y]$$

by the definition of Π^+ in 3.15. Observing now that

$$(x, a^{-1}, x^{-1}, x, a^2, x^{-1}) \in \mathbf{D},$$

via $N_S(U)^{x^{-1}}$, we obtain

$$[b] = \Pi^+(w) = [y^{-1}, h^{-1}(xax^{-1})h, y] = [y^{-1}, (a^{x^{-1}})^h, y].$$

We now claim that $(y^{-1}, b^{y^{-1}}, y) \in [b]$. In order to see this, one observes first of all that $(y^{-1}, b^{y^{-1}}, y) \in \Phi$. Also, since $b^{y^{-1}}$ normalizes $N_S(V)^{y^{-1}}$ we have $(y^{-1}, y, b, y^{-1}, y) \in \mathbf{D}$ via $N_S(V)$. Then $\Pi(y^{-1}, b^{y^{-1}}, y) = b$, and the claim is proved. Thus:

$$(**) \qquad [y^{-1}, (a^{x^{-1}})^h, y] = [y^{-1}, b^{y^{-1}}, y].$$

Application of Π to both sides of (**) then yields

$$((a^{x^{-1}})^h)^y = (b^{y^{-1}})^y,$$

and then $(a^{x^{-1}})^h = b^{y^{-1}}$ by the cancellation rule in $\mathcal{L}$. As $a^{x^{-1}}, b^{y^{-1}} \in S$ by (*), we conclude that $a \in S_\phi$. This completes the proof of (a), and thereby completes the proof. □

At this point it will be convenient (and need cause no confusion) to view λ_0 and λ as inclusion maps. Then $\mathcal{L}^+ = \mathcal{L} \cup \mathcal{L}_1^+$, where

$$\mathcal{L}_1^+ = \{[\phi] \mid \phi \in \Phi,\ \phi \notin \mathbf{D}\}.$$

Proposition 3.17. *$(\mathcal{L}^+, \Delta^+, S)$ is a locality. Moreover:*

(a) *$\mathcal{L}$ is the restriction $\mathcal{L}^+ |_\Delta$ of $\mathcal{L}^+$ to Δ.*
(b) *$N_{\mathcal{L}^+}(R) = N_{\mathcal{L}}(R)$.*
(c) *$\mathcal{F}_S(\mathcal{L}^+) = \mathcal{F}$.*

Proof. We have first to show that $(\mathcal{L}^+, \Delta^+)$ satisfies the conditions (O1) and (O2) in the definition I.2.1 of objectivity. Condition (O2) is the requirement that Δ^+ be $\mathcal{F}$-closed (i.e. that Δ^+ be preserved by $\mathcal{F}$-homomorphisms). Since Δ is $\mathcal{F}$-closed, and Δ^+ is given by attaching to Δ an $\mathcal{F}$-conjugacy class $R^{\mathcal{F}}$ and all overgroups in S of members of $R^{\mathcal{F}}$, (O2) holds for $(\mathcal{L}^+, \Delta^+)$.

Condition (O1) requires that $\mathbf{D}^+$ be equal to $\mathbf{D}_{\Delta^+}$. This means:

(*) The word $w = (C_1, \cdots, C_n) \in \mathbf{W}(\mathcal{L}^+)$ is in $\mathbf{D}^+ \iff$ there exists a sequence $(X_0, \cdots, X_n) \in \mathbf{W}(\Delta^+)$ such that $X_{i-1}^{C_i} = X_i$ for all i, $(1 \leq i \leq n)$.

Here we need only be concerned with the case $w \in \mathbf{W}(\mathcal{L}_0^+)$ since $\mathbf{D}^+ = \mathbf{D}_0^+ \cup [\mathbf{D}]$, and since $\mathbf{D} = \mathbf{D}_\Delta$. The implication $\implies$ in (*) is then given by the definition of $\mathbf{D}_0^+$, with $X_0 \in R^{\mathcal{F}}$. The reverse implication is given by 3.16, and thus $(\mathcal{L}^+, \Delta^+)$ is objective. We note that $\mathcal{L}^+$ is finite since $\mathcal{L}$ and Φ are finite.

Let $\widetilde{S}$ be a p-subgroup of $\mathcal{L}^+$ containing S, and let $a \in N_{\widetilde{S}}(S)$. As $S \in \Delta$ we get $a \notin \mathcal{L}_1^+$, so $a \in N_{\mathcal{L}}(S)$, and then $a \in S$ since S is a maximal p-subgroup of $\mathcal{L}$. Thus $\widetilde{S} = S$, S is a maximal p-subgroup of $\mathcal{L}^+$, and $(\mathcal{L}^+, \Delta^+, S)$ is a locality. The restriction of $\mathcal{L}^+$ to Δ is by definition the partial group whose product is the restriction of Π^+ to $\mathbf{D}_\Delta$, whose underlying set is the image of $\Pi^+ \mid \mathbf{D}_\Delta$, and whose inversion map is inherited from $\mathcal{L}^+$. That is, (a) holds.

Let $[\phi] \in \mathcal{L}_1^+$, let $\phi = (x^{-1}, h, y) \in [\phi]$, and set $U = U\phi$. Then $U \trianglelefteq S_{[\phi]} = S_\phi$ by 4.17(a), and the conjugation map $c_{[\phi]} : S_\phi \to S$ is then the composite $c_x^{-1} \circ c_h \circ c_y$ applied to S_ϕ. Thus $c_{[\phi]}$ is an $\mathcal{F}$-homomorphism, and this yields (c). Suppose now that $[\phi] \in N_{\mathcal{L}^+}(R)$. Then x, h, and y are in $N_{\mathcal{L}}(R)$, and then $\phi \in \mathbf{D}$ as $N_{\mathcal{L}}(R)$ is a subgroup of $\mathcal{L}$. Then $[\phi] \notin \mathcal{L}_1^+$, and so (b) holds. $\square$

Proposition 3.18. *Let $(\widetilde{\mathcal{L}}, \Delta^+, S)$ be a locality having the same set Δ^+ of objects as $\mathcal{L}^+$. Assume that $\widetilde{\mathcal{L}} \mid_\Delta = \mathcal{L}$ and that $N_{\widetilde{\mathcal{L}}}(R) = N_{\mathcal{L}}(R)$. Then the identity map on $\mathcal{L}$ extends in a unique way to an isomorphism $\mathcal{L}^+ \to \widetilde{\mathcal{L}}$.*

Proof. Write $\widetilde{\Pi} : \widetilde{\mathbf{D}} \to \widetilde{\mathcal{L}}$ for the product in $\widetilde{\mathcal{L}}$. (It isn't necessary to distinguish the inversion map on $\widetilde{\Pi}$ in any way, since by I.1.13 it restricts to the inversion map on $\mathcal{L}$.) Let $\phi = (x^{-1}, h, y)$ and $\overline{\phi} = (\overline{x}^{-1}, \overline{h}, \overline{y})$ be members of Φ such that $\phi \sim \overline{\phi}$. Set $U = U_\phi$. Then $U = U_{\overline{\phi}}$, and $(\overline{x}, x^{-1}) \in \mathbf{D}$ via $N_S(U)^{\overline{x}^{-1}}$. Then $\widetilde{\Pi}(\overline{x}, x^{-1}) = \Pi(\overline{x}, x^{-1})$ as $\mathcal{L}$ is the restriction of $\widetilde{\mathcal{L}}$ to Δ. Similarly, we obtain $\widetilde{\Pi}(\overline{y}, y^{-1}) = \Pi(\overline{y}, y^{-1})$. Then:

$$(1) \qquad \widetilde{\Pi}(\overline{x}, x^{-1}, h) = \widetilde{\Pi}(\overline{x}x^{-1}, h) = \Pi(\overline{x}x^{-1}, h) = \Pi(\overline{h}, \overline{y}y^{-1}) = \widetilde{\Pi}(\overline{h}, \overline{y}, y^{-1}).$$

Observe that both $(\overline{x}^{-1}, \overline{x}, x^{-1}, h, y)$ and $(\overline{x}^{-1}, \overline{h}, \overline{y}, y^{-1}, y)$ are in $\mathbf{D}^+ \cap \widetilde{\mathbf{D}}$ (via the obvious conjugates of U). Then (1) yields

$$\widetilde{\Pi}(\phi) = \widetilde{\Pi}(\overline{x}^{-1}, \overline{x}, x^{-1}, h, y) = \widetilde{\Pi}(\overline{x}^{-1}, \overline{h}, \overline{y}, y^{-1}, y) = \widetilde{\Pi}(\overline{\phi}).$$

If $\phi \in \mathbf{D}$ then $\widetilde{\Pi}(\phi) = \Pi(\phi)$, so we have shown that there is a well-defined mapping

$$\beta : \mathcal{L}^+ \to \widetilde{\mathcal{L}}$$

such that β restricts to the identity map on $\mathcal{L}$ and sends the element $[\phi] \in \mathcal{L}_0^+$ to $\widetilde{\Pi}(\phi)$.

We now show that β is a homomorphism of partial groups. As $\mathcal{L}^+$ is a pushout, in the manner described in 3.15(c), and since the inclusion $\mathcal{L} \to \widetilde{\mathcal{L}}$ is a homomorphism, it suffices to show that the restriction β_0 of β to $\mathcal{L}_0^+$ is a homomorphism.

So, let $w = ([\phi_1], \cdots, [\phi_n]) \in \mathbf{D}^+$, let $\gamma = (\phi_1, \cdots, \phi_n)$ be a Γ-form of w, and set $w_\gamma = (\phi_1, \cdots, \phi_n)$. Write $\phi_i = (x_i^{-1}, h_i, y_i)$. Then $\Pi^+(w) = [x_1^{-1}, \Pi(w_0), y_n]$, where $w_0 \in \mathbf{W}(N_{\mathcal{L}}(R))$ is the word

$$w_0 = (h_1, y_1 x_2^{-1}, \cdots, y_{n-1} x_n^{-1}, h_n).$$

given by 3.12(b). Let $\beta*$ be the induced mapping $\mathbf{W}(\mathcal{L}^+) \to \mathbf{W}(\widetilde{\mathcal{L}})$ of free monoids. Then

$$\widetilde{\Pi}(w\beta^*) = \widetilde{\Pi}([\phi_1]\beta, \cdots, [\phi_n]\beta) = \widetilde{\Pi}(\widetilde{\Pi}(\phi_1), \cdots, \widetilde{\Pi}(\phi_n)) = \widetilde{\Pi}(w_\gamma)$$

by $\widetilde{\mathbf{D}}$-associativity, and

$$(\Pi^+(w))\beta = [x_1^{-1}, \Pi(w_0), y_n]\beta = \widetilde{\Pi}(x_1^{-1}, \Pi(w_0), y_n).$$

Set $w_0' = (h_1, y_1, x_2^{-1}, \cdots, y_{n-1}, x_n^{-1}, h_n)$. Then $w_0' \in \widetilde{\mathbf{D}}$ via U_{ϕ_1}, and $\widetilde{\Pi}(w_0') = \widetilde{\Pi}(w_0) = \Pi(w_0)$. Then

$$(\Pi^+(w))\beta = \widetilde{\Pi}((x_1^{-1}) \circ \widetilde{\Pi}(w_0') \circ (y_n)) = \widetilde{\Pi}(w_\gamma) = \widetilde{\Pi}(w\beta^*),$$

and so β_0 is a homomorphism. As already mentioned, this result implies that β is a homomorphism.

Let $f \in \widetilde{\mathcal{L}}$ with $f \notin \mathcal{L}$. Then $S_f \notin \Delta$ (as $\mathcal{L} = \widetilde{\mathcal{L}} \mid_\Delta$), and so S_f contains a unique $U \in R^{\mathcal{F}}$. Set $V = U^f$, and recall that the inversion map on $\mathcal{L}$ is induced from the inversion map on $\widetilde{\mathcal{L}}$. If $V \in \Delta$ then $f^{-1} \in \widetilde{\mathcal{L}}$, and then $f \in \mathcal{L}$. Thus $V \notin \Delta$, and hence $V \in R^{\mathcal{F}}$. By 3.1 there exist elements $x, y \in \mathcal{L}$ such that $U^x = R = V^y$ and such that $N_S(U)^{x^{-1}} \leq N_S(R) \geq N_S(V)^{y^{-1}}$. We find that $(x, f, y^{-1}) \in \widetilde{\mathbf{D}}$ via R and that $h := \widetilde{\Pi}(x, f, y^{-1}) \in N_{\widetilde{\mathcal{L}}}(R)$. Then $h \in N_{\mathcal{L}}(R)$ by hypothesis, and $f = \widetilde{\Pi}(x^{-1}, h, y)$. In particular, we have shown:

(*) Every element of $\widetilde{\mathcal{L}}$ is a product of elements of $\mathcal{L}$.

We may now show that β^* maps $\mathbf{D}_0^+$ onto $\widetilde{\mathbf{D}}$. Thus, let $\widetilde{w} = (f_1, \cdots, f_n) \in \widetilde{\mathbf{D}}$ and set $X = S_{\widetilde{w}}$. If $X \in \Delta$ then $\widetilde{w} \in \mathbf{D}$ and $\widetilde{w} = \widetilde{w}\beta^*$. So assume that $X \notin \mathbf{D}$. Then X contains a unique $\mathcal{F}$-conjugate U_0 of R, and there is a sequence $(U_1, \cdots, U_n) \in \mathbf{W}(\Delta^+)$ given by $U_i = (U_{i-1})_i^f$. As seen in the preceding paragraph, we then have $U_i \in R^{\mathcal{F}}$ for all i, and there exists a sequence $\gamma = (\phi_1, \cdots, \phi_n) \in \mathbf{W}(\Phi)$ such that $\widetilde{\Pi}(\phi_i) = f_i$. Set $w_\gamma = [\phi_1] \circ \cdots \circ [\phi_n]$. Then $w_\gamma \in \mathbf{D}^+$ and $(w_\gamma)\beta^* = \widetilde{w}$. Thus $\beta^* : \mathbf{D}_0^+ \to \widetilde{\mathbf{D}}$ is surjective. That is, β is a projection, as defined in I.4.5.

Let $g \in Ker(\beta)$. If $g \in \mathcal{L}$ then $g = \mathbf{1}$ since $\beta \mid_{\mathcal{L}}$ is the inclusion map. So assume that $g \notin \mathcal{L}$. Then $g = [\phi]$ for some $\phi = (x^{-1}, h, y) \in \Phi$, and $[\phi]$ is a $\sim$-class. Set $U = U_\phi$ and

$V = V_\phi$. As $\mathbf{1} = g\beta = \widetilde{\Pi}(\phi)$ it follows that $U = V = R$. But then $\phi \in \mathbf{D}$ as $N_{\mathcal{L}}(R)$ is a subgroup of $\mathcal{L}$, and so $[\phi]$ is not a $\sim$-class. Thus $Ker(\beta) = \mathbf{1}$. As β is a projection, β is then an isomorphism by I.4.3(d). It follows from (*) that β is the unique isomorphism $\mathcal{L}^+ \to \widetilde{\mathcal{L}}$ which restricts to the identity on $\mathcal{L}$. $\square$

Notice that Propositions 3.17 and 3.18 complete the proof of Theorem 3.3.

Section 4: Elementary expansions and partial normal subgroups

We continue to assume Hypothesis 3.2 throughout this section. Our aim is to prove the following result.

Theorem 4.1. *Assume Hypothesis 3.2 and let $(\mathcal{L}^+, \Delta^+, S)$ be the locality given by Theorem 3.3. Let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup of $\mathcal{L}$, and let $\mathcal{N}^{\mathcal{L}^+}$ be the set of all $\mathcal{L}^+$-conjugates of elements of $\mathcal{N}$ (the set of all f^g such that $f \in \mathcal{N}$, $g \in \mathcal{L}^+$ and $(g^{-1}, f, g) \in \mathbf{D}^+$). Let $\mathcal{N}^+$ be the smallest partial subgroup of $\mathcal{L}^+$ containing $\mathcal{N}^{\mathcal{L}^+}$. Then*

(a) *$\mathcal{N}^+ \trianglelefteq \mathcal{L}^+$, and*

(b) *$\mathcal{N}^+ \cap \mathcal{L} = \mathcal{N}$.*

We will employ all of the notation from section 3. Thus, the reader will need to have in mind the meanings of $\mathcal{L}_0$, $\mathcal{L}_0^+$, $\mathcal{L}^+$, Φ, $\sim$, and $\approx$. Set $\Omega = \mathcal{N}^{\mathcal{L}^+}$.

Lemma 4.2. *Let $w \in \mathbf{D}^+$, and suppose that $w \notin \mathbf{W}(\mathcal{L})$. Then S_w is an $\mathcal{F}$-conjugate of R, and $S_w = S_{\Pi^+(w)}$ if $\Pi^+(w) \notin \mathcal{L}$.*

Proof. As $w \notin \mathcal{L}$ it follows that $S_w \notin \Delta$. Then, as $w \in \mathcal{L}^+$, it follows from 3.2(1) that S_w is an $\mathcal{F}$-conjugate of R. Write $w = (g_1, \cdots, g_n)$ with $g_i \in \mathcal{L}^+$, set $g = \Pi^+(w)$, and suppose that $g \notin \mathcal{L}$. Then S_g is an $\mathcal{F}$-conjugate, and since $S_w \leq S_g$ we obtain $S_w = S_g$, as required. $\square$

Lemma 4.3. *If 4.1(b) holds, then Theorem 4.1 holds in its entirety.*

Proof. Assume 4.1(b). Set $\Omega_0 = \Omega$, and recursively define Ω_n for $n > 0$ by

$$\Omega_n = \{\Pi^+(w) \mid w \in \mathbf{W}(\Omega_{n-1}) \cap \mathbf{D}^+\}.$$

As $\mathcal{N}^+ = \langle \Omega \rangle$ is by definition the smallest partial subgroup of $\mathcal{L}^+$ containing Ω, it follows from I.1.9 that $\mathcal{N}^+$ is the union of the sets Ω_n. In order to show that $\mathcal{N}^+ \trianglelefteq \mathcal{L}^+$ it will then suffice to show that each Ω_n is closed with respect to conjugation in $\mathcal{L}^+$.

Let $g \in \Omega_0$. Then there exists $f \in \mathcal{N}$ and $a \in \mathcal{L}^+$ such that $(a^{-1}, f, a) \in \mathbf{D}^+$ and with $g = f^a$. Now let $b \in \mathcal{L}^+$ such that $(b^{-1}, g, b) \in \mathbf{D}^+$. If $g \in \mathcal{L}$ then $g \in \mathcal{N}$ by 4.1(b), and then $g^b \in \Omega_0$. On the other hand, assume $g \notin \mathcal{L}$. Then $S_g \in R^{\mathcal{F}}$, and 4.2 yields $S_g = S_{(a^{-1}, f, a)}$. Set $u = (b^{-1}, g, b)$ and $v = (b^{-1}, a^{-1}, f, a, b)$. Then $S_u = S_v = (S_g)^b$, so $v \in \mathbf{D}^+$, and $g^b = \Pi^+(u) = \Pi^+(v) = f^{ab}$. Thus $g^b \in \Omega_0$ in any case, and so Ω_0 is closed with respect to conjugation in $\mathcal{L}^+$.

Let k be the largest index (assuming it exists) such that Ω_k is closed with respect to conjugation in $\mathcal{L}^+$, and let now $f \in \Omega_{k+1}$ and $b \in \mathcal{L}$ such that f^b is defined in $\mathcal{L}^+$ and

is not in Ω_{k+1}. By definition there exists $w = (f_1, \cdots, f_n) \in \mathbf{W}(\Omega_k) \cap \mathbf{D}^+$ such that $f = \Pi^+(w)$. Suppose $f \in \mathcal{L}$. Then $f \in \mathcal{N}$ by 4.1(b), and then $f^b \in \Omega$. As $\Omega \subseteq \Omega_m$ for all m, we have $f^b \in \Omega_{k+1}$ in this case. So assume that $f \notin \mathcal{L}$. Then $S_f = S_w \in R^{\mathcal{F}}$ by 4.2. Set $w' = (b^{-1}, f_1, b, \cdots, b^{-1}, f_n, b)$. Then $w' \in \mathbf{D}^+$ via $(S_w)^b$, and we obtain $f^b = \Pi^+(w') = \Pi^+(f_1^b, \cdots, f_n^b)$. As Ω_k is closed with respect to conjugation in $\mathcal{L}^+$ we conclude that $f^b \in \Omega_{k+1}$, contrary to the maximality of k; and this contradiction completes the proof. $\square$

(4.4) Notation. Set $M = N_{\mathcal{L}}(R)$, and $K = M \cap \mathcal{N}$. As in the preceding section, for each $g \in \mathcal{L}^+$ set

$$\mathcal{U}_g = \{U \in R^{\mathcal{F}} \mid U \leq S_g\},$$

and for each $U \in R^{\mathcal{F}}$ set

$$\mathbf{Y}_U = \{y \in \mathcal{L} \mid R^y = U,\ N_S(U)^{y^{-1}} \leq N_S(R)\}.$$

The notation (4.5) will remain fixed until the proof of Theorem 5.2 is complete. Note that since M is a subgroup of $\mathcal{L}$ by 3.2, K is a normal subgroup of M by I.1.8(c).

Lemma 4.6. *Suppose $T \leq R$. Then $\mathcal{N}^* = \Omega$, and $\Omega \cap \mathcal{L} = \mathcal{N}$.*

Proof. Let $f \in \mathcal{N}$, and suppose that S_f contains an $\mathcal{F}$-conjugate of R. Then $T \leq S_f$ since, by I.3.1(a) T is weakly closed in $\mathcal{F}$. Then I.3.1(b) yields the following result.

(1) Let $f \in \mathcal{N}$ such that S_f contains an $\mathcal{F}$-conjugate U of R. Then $P^f = P$ for each subgroup P of S_f containing U. In particular, $U^f = U$.

Let $f' \in \Omega$, and let $f \in \mathcal{N}$ and $g \in \mathcal{L}^+$ such that $f' = f^g$. That is, assume that $v := (g^{-1}, f, g) \in \mathbf{D}^+$ and that $f' = \Pi^+(v)$. If $v \in \mathbf{D}$ then $f' = \Pi(v) \in \mathcal{N}$. On the other hand, suppose that $v \notin \mathbf{D}$. Then S_v is an $\mathcal{F}$-conjugate of R by 4.1(1). Set $U = (S_v)^{g^{-1}}$. Then $U = U^f$ by (1). Now choose $a \in \mathbf{Y}_U$, and set $v' = (g^{-1}, a^{-1}, a, f, a^{-1}, a, g)$. Then $v' \in \mathbf{D}^+$ via U^g, and $\Pi^+(v) = \Pi^+(v')$. Notice that (1) implies that $(a, f, a^{-1}) \in \mathbf{D}$ via $P := N_{S_f}(U)$, and that

$$T \xrightarrow{a} U \xrightarrow{f} U \xrightarrow{a^{-1}} T,$$

so that $f^{a^{-1}} \in M$. Thus $f^{a^{-1}} \in K$, and $\Pi^+(v) = \Pi^+(g^{-1}a^{-1}, f^{a^{-1}}, ag)$. This shows:

(2) Ω is the union of $\mathcal{N}$ with the set of all $\Pi^+(g^{-1}, f, g)$ such that $f \in K$ and such that $(g^{-1}, f, g) \in \mathbf{D}^+$. Moreover, for any such $v = (g^{-1}, f, g)$, $\Pi^+(v)$ normalizes each $V \in R^{\mathcal{F}}$ such that $V \leq S_v$.

Assume now that we have $v = (g^{-1}, f, g)$ as in (2) (so that $f \in K$), and let A be an $\mathcal{F}$-conjugate of R contained in S_v. In order to analyze these things further, we shall need to be able to compute products in $\mathcal{L}^+$ in the manner described in 4.13 and 4.14. To that end, note first of all that since $A^g = T$ there exists a unique $h \in M$ and $y \in \mathbf{Y}_A$ such that $g \approx (\mathbf{1}, h, y) \in \Phi$, by 4.7. Set $\phi = (\mathbf{1}, h, y)$ and set $\psi = (\mathbf{1}, f, \mathbf{1})$. Then (ϕ^{-1}, ψ, ϕ) is a Γ-form of (g^{-1}, f, g), as defined in 4.12. We then compute via 4.13 that

$$f' = \Pi^+(g^{-1}, f, g) = [y^{-1}, h^{-1}, \mathbf{1}][\mathbf{1}, f, \mathbf{1}][\mathbf{1}, h, y] = [y^{-1}, f^h, y].$$

(3) Let $f' \in \Omega$, such that $S_{f'}$ contains an $\mathcal{F}$-conjugate of R. Then f' is an $\approx$-class $[y^{-1}, k, y]$ with $k \in K$.

If $f' \in \mathcal{L}$ then $(y^{-1}, k, y) \in \mathbf{D}$ by 4.10, and so $f' \in \mathcal{N}$. Thus:

(4) $\Omega \cap \mathcal{L} = \mathcal{N}$.

Now let $w = (f_1', \cdots, f_n') \in \mathbf{W}(\Omega) \cap \mathbf{D}^+$, and set $B = S_w$. Suppose that $\Pi^+(w) \notin \Omega$. Then $\Pi^+(w) \notin \mathcal{N}$, so $\Pi^+(w) \notin \mathcal{L}$ by (4). Thus $w \notin \mathbf{D}$, so $B \in R^{\mathcal{F}}$, and then (2) shows that each f_i' normalizes B. Fix $b \in \mathbf{Y}_B$. Then (3) implies that there exist elements $k_i \in K$ such that $f_i' = [b^{-1}, k_i, b]$. One observes that the sequence of elements (b^{-1}, k_i, b) of Φ is a Γ-form for w, and then 4.13 yields $\Pi^+(w) = [b^{-1}, k, b]$ where $k = \Pi(k_1, \cdots, k_n) \in K$. This simply means that $\Pi^+(w) = k^b$, since $b^{-1} = [b^{-1}, \mathbf{1}, \mathbf{1}]$, $k = [\mathbf{1}, k, \mathbf{1}]$, and $b = [\mathbf{1}, \mathbf{1}, b]$. Thus $\Pi^+(w) \in \Omega$. Since Ω is closed under inversion, we have thus shown that Ω is a partial subgroup of $\mathcal{L}$.

Finally, let $c \in \mathcal{L}^+$ be given so that $(c^{-1}, f', c) \in \mathbf{D}^+$ (and where $f' \in \Omega$). Suppose that $\Pi^+(c^{-1}, f', c) \notin \Omega$. Then $f' \notin \mathcal{N}$, so $f' \notin \mathcal{L}$ by (4), and then $f' = \Pi^+(g^{-1}, f, g)$ for some $f \in \mathcal{N}$ and some $g \in \mathcal{L}^+$, and where $S_{f'} = S_{(g^{-1}, f, g)}$. Set $u = (c^{-1}, g^{-1}, f, g, c)$. Then $u \in \mathbf{D}^+$ via $(S_{f'})^c$, and $\Pi^+(u) = \Pi^+(c^{-1}g^{-1}, f, gc)$. Thus $\Pi^+(c^{-1}, f', c) \in \Omega$ after all, and $\Omega \trianglelefteq \mathcal{L}^+$. $\square$

Let $\overline{L}$ be the quotient locality $\mathcal{L}/\mathcal{N}$ (cf. 4.4), and let $\rho : \mathcal{L} \to \overline{\mathcal{L}}$ be the quotient map and let ρ^* be the induced homomorphism $\mathbf{W}(\mathcal{L}) \to \mathbf{W}(\overline{\mathcal{L}})$ of free monoids. For any subset or element X of $\mathcal{L}$, $\overline{X}$ shall denote the image of X under ρ. We extend this convention to subsets and elements of $\mathbf{W}(\overline{\mathcal{L}})$ in the obvious way. Set $\overline{\Delta} = \{\overline{P} \mid P \in \Delta\}$.

Lemma 4.7. *Assume that $T \nleq R$, and set $\overline{\mathcal{F}} = \mathcal{F}_{\overline{S}}(\overline{\mathcal{L}})$. Then $\overline{R}^{\overline{\mathcal{F}}} \subseteq \overline{\Delta}$.*

Proof. Let $U \in R^{\mathcal{F}}$. As $T \nleq R$ we then have $T \nleq U$ by I.3.1(a). Then U is a proper subgroup of UT, so $UT \in \Delta$ by 3.2(1). Then $\overline{U} = \overline{UT} \in \overline{\Delta}$ by 4.3. $\square$

Lemma 4.8. *Assume that $T \nleq R$. There is then a homomorphism $\sigma : \mathcal{L}^+ \to \overline{\mathcal{L}}$ such that the restriction of σ to $\mathcal{L}$ is the quotient map ρ.*

Proof. Set $\overline{\mathbf{D}} = \mathbf{D}(\overline{\mathcal{L}})$ and let $\overline{\Pi} : \overline{\mathbf{D}} \to \overline{\mathcal{L}}$ be the product in $\overline{\mathcal{L}}$. As $\overline{R}^{\overline{\mathcal{F}}} \subseteq \overline{\Delta}$ by 4.7, we have $\overline{\Phi} \subseteq \overline{\mathbf{D}}$, and so there is a mapping $\lambda : \Phi \to \overline{\mathcal{L}}$ given by $\phi\lambda = \overline{\Pi}(\phi\rho^*)$, where $\rho^* : \mathbf{W}(\mathcal{L}) \to \mathbf{W}(\overline{\mathcal{L}})$ is the homomorphism of free monoids induced by ρ. That is: $\phi\lambda = \overline{\Pi}(\overline{\phi})$.

Let $\phi_1, \phi_2 \in \Phi$ with $\phi_1 \sim \phi_2$, and write $\phi_i = (x_i^{-1}, h_i, y_i)$. Then $(x_2 x_1^{-1})h_1 = h_2(y_2 y_1^{-1})$ (in M), and so

$$(\overline{x}_2 \overline{x}_1^{-1})\overline{h}_1 = \overline{h}_2(\overline{y}_2 \overline{y}_1^{-1}) \tag{1}$$

in $\overline{M}$. As (x_2, x_1^{-1}, h_1) and (h_2, y_2, y_1^{-1}) are in $\mathbf{D}^+$ via the appropriate conjugates of R, we have $(\overline{x}_2, \overline{x}_1^{-1}, \overline{h}_1)$ and $(\overline{h}_2, \overline{y}_2, \overline{y}_1^{-1})$ in $\overline{\mathbf{D}}$, and then

$$\overline{\Pi}(\overline{x}_2, \overline{x}_1^{-1}, \overline{h}_1) = \overline{\Pi}(\overline{h}_2, \overline{y}_2, \overline{y}_1^{-1})$$

by (1) and $\overline{\mathbf{D}}$-associativity. A standard cancellation argument (cf. I.2.4(a)) then yields $\overline{\Pi}(\overline{\phi}_1) = \overline{\Pi}(\overline{\phi}_2)$, and thus λ is constant on $\sim$-classes.

Now suppose that $\phi \in \Phi_g$ for some $g \in \mathcal{L}$. That is, suppose that $\phi \in \Phi \cap \mathbf{D}$ and that $g = \Pi(\phi)$. Then $\phi\lambda = \overline{g}$ since ρ is a homomorphism of partial groups, and so λ is constant on $\approx$-classes. This shows that there is a (well-defined) mapping $\sigma : \mathcal{L}^+ \to \overline{\mathcal{L}}$ given by ρ on $\mathcal{L}$ and by λ on $\mathcal{L}_0^+$. It remains only to check that σ is a homomorphism.

Let $w = (f_1, \cdots, f_n) \in \mathbf{D}^+$. If $w \in \mathbf{D}$ then $w\sigma^* = w\rho^* \in \overline{\mathbf{D}}$ and $\overline{\Pi}(w\sigma^*) = (\Pi(w))\sigma$. On the other hand, suppose that $w \notin \mathbf{D}$. Then $f_i = [\phi_i]$ for some $\phi_i \in \Phi$, and there is a Γ-form $\gamma = (\phi_1, \cdots, \phi_n)$ of w. Thus the word $w_\gamma = \phi_1 \circ \cdots \circ \phi_n$ has the property that S_{w_γ} is an $\mathcal{F}$-conjugate of R, and hence $\overline{w_\gamma} \in \overline{\mathbf{D}}$. Then

$$\overline{\Pi}((w_\gamma)\sigma^*) = \overline{\Pi}(\overline{w_\gamma}).$$

Write $\phi_i = (x_i^{-1}, h_i, y_i)$. Then

$$(\Pi^+(w_\gamma))\sigma = [x_1^{-1}, \Pi(w_0), y_n]\sigma$$

where $w_0 \in \mathbf{D}(M)$ is given by the formula in 4.13(b). One then obtains $\overline{\Pi}((w_\gamma)\sigma^*) = (\Pi^+(w_\gamma))\sigma$ via $\overline{\mathbf{D}}$-associativity, and so σ is a homomorphism, as desired. $\square$

Lemma 4.9. $\mathcal{N}^* \cap \mathcal{L} = \mathcal{N}$.

Proof. Let $\sigma : \mathcal{L}^+ \to \overline{\mathcal{L}}$ be a homomorphism, as in 4.8, whose restriction to $\mathcal{L}$ is the quotient map $\rho : \mathcal{L} \to \overline{\mathcal{L}}$. Then $Ker(\sigma) \cap \mathcal{L} = Ker(\rho)$, where $Ker(\rho) = \mathcal{N}$, by 4.4. As $Ker(\sigma) \trianglelefteq \mathcal{L}^+$, by 1.14, the lemma follows. $\square$

Notice that Theorem 4.1 follows at once from the combination of lemmas 4.3, 4.6, and 4.9.

Section 5: Theorem A

Recall from 1.8 that for any fusion system $\mathcal{F}$ on S, $\mathcal{F}^s$ denotes the set of $\mathcal{F}$-subcentric subgroups of S, and that these are the subgroups $U \le S$ such that there exists an $\mathcal{F}$-conjugate V of U with V fully normalized in $\mathcal{F}$ and with $O_p(N_\mathcal{F}(V)) \in \mathcal{F}^c$.

Lemma 5.1. *Let $(\mathcal{L}, \Delta, S)$ be a proper locality on $\mathcal{F}$, and let R be a subgroup of S which satisfies 3.2(1) and 3.2(2). Assume that $R \in \mathcal{F}^s$. Then R satisfies 3.2(3). That is, $N_\mathcal{L}(R)$ is a subgroup of $\mathcal{L}$ of characteristic p, and $N_\mathcal{F}(R) = \mathcal{F}_{N_S(R)}(N_\mathcal{L}(R))$.*

Proof. Set $\mathcal{L}_R = N_\mathcal{L}(R)$ and set $\Delta_R = \{P \in \Delta \mid R \trianglelefteq P\}$. By 3.1 $(\mathcal{L}_R, \Delta_R, N_S(R))$ is a locality, and we set $\mathcal{F}_R = \mathcal{F}_{N_S(R)}(\mathcal{L}_R)$. Evidently $\mathcal{F}_R$ is a fusion subsystem of $N_\mathcal{F}(R)$.

If $R \in \Delta$ then $\mathcal{L}_R$ is a subgroup of $\mathcal{L}$, of characteristic p since $\mathcal{L}$ is proper; and $N_\mathcal{F}(R) = \mathcal{F}_{N_S(R)}(\mathcal{L}_R)$ by 2.2. Thus there is nothing to show in this case, and so we may assume that $R \notin \Delta$. As $R \in \mathcal{F}^s$ there exists an $\mathcal{F}$-conjugate R' of R such that $O_p(N_\mathcal{F}(R'))$ is centric in $\mathcal{F}$. As $\mathcal{F}$ is $\Delta \cup R^\mathcal{F}$-inductive by 3.1(c), it follows from 1.14(a) that $Q := O_p(N_\mathcal{F}(R))$ is centric in $\mathcal{F}$. If $R = Q$ then $R \in \mathcal{F}^{cr}$, contrary to $R \notin \Delta$. Thus

$R \neq Q$ and so $Q \in \Delta$ by 3.2(1). As $\mathcal{F}_R$ is a fusion subsystem of $N_{\mathcal{F}}(R)$, on $N_S(R)$, we have $Q \trianglelefteq \mathcal{F}_R$, and then Q is contained in every member of $(\mathcal{F}_R)^{cr}$. Thus:

(*) $(\mathcal{F}_R)^{cr} \subseteq \Delta_R$.

For $P \in \Delta_R$ write $\mathcal{L}_P$ for the subgroup $N_{\mathcal{L}}(P)$ of $\mathcal{L}$. Then $R \leq \mathcal{L}_P$, and and $N_{\mathcal{L}_P}(R) = N_{\mathcal{L}_R}(P)$. As $\mathcal{L}$ is proper, $\mathcal{L}_P$ is of characteristic p, and $N_{\mathcal{L}_R}(P)$ is then of characteristic p by II.2.7(b). This result, along with (*), shows that $\mathcal{L}_R$ is a proper locality on $\mathcal{F}_R$. Then $Q \trianglelefteq \mathcal{L}_R$ by 2.3, and thus $\mathcal{L}_R = N_{\mathcal{L}_Q}(R)$ is a subgroup of $\mathcal{L}$ of characteristic p.

It now remains only to show that $\mathcal{F}_R = N_{\mathcal{F}}(R)$, in order to complete the proof. Observe that $N_{\mathcal{F}}(R)$ is a fusion subsystem of $N_{\mathcal{F}}(Q)$, and that $N_{\mathcal{F}}(Q)$ is the fusion system $\mathcal{F}_{N_S(Q)}(\mathcal{L}_Q)$ of a finite group, by 2.2(a). Then each $N_{\mathcal{F}}(R)$-isomorphism is a conjugation map by an element of $N_{\mathcal{L}_Q}(R)$, and so $N_{\mathcal{F}}(R)$ is a fusion subsystem of $\mathcal{F}_R$. That $\mathcal{F}_R$ is a subsystem of $N_{\mathcal{F}}(R)$ has already been noted, so the required equality of fusion systems obtains. $\square$

5.2 (Theorem A1). *Let $(\mathcal{L}, \Delta, S)$ be a proper locality on $\mathcal{F}$ and let Δ^+ be an $\mathcal{F}$-closed collection of subgroups of S such that $\Delta \subseteq \Delta^+ \subseteq \mathcal{F}^s$.*

(a) *There exists a proper locality $(\mathcal{L}^+, \Delta^+, S)$ on $\mathcal{F}$ such that $\mathcal{L}$ is the restriction $\mathcal{L}^+ \mid_\Delta$ of $\mathcal{L}^+$ to Δ. Moreover, $\mathcal{L}^+$ is generated by $\mathcal{L}$ as a partial group.*

(b) *For any proper locality $(\widetilde{\mathcal{L}}, \Delta^+, S)$ on $\mathcal{F}$ whose restriction to Δ is $\mathcal{L}$, there is a unique isomorphism $\mathcal{L}^+ \to \widetilde{\mathcal{L}}$ which restricts to the identity map on $\mathcal{L}$.*

Proof. Suppose false, and among all counterexamples choose $(\mathcal{L}, \Delta, S)$ and Δ^+ so that the set $\mathcal{U} = \Delta^+ - \Delta$ has the smallest possible cardinality. Then $\Delta^+ = \Delta \cup R^{\mathcal{F}}$ for some $R \in \Delta^+$. By 1.15 R may be chosen so that both R and $O_p(N_{\mathcal{F}}(R))$ are fully normalized in $\mathcal{F}$. Thus the conditions (1) and (2) in Hypothesis 3.2 hold. Then 3.2 holds in its entirety, by 5.1.

Set $\Delta_1 = \Delta \bigcup R^{\mathcal{F}}$. Theorem 3.4 then yields a proper locality $(\mathcal{L}_1, \Delta_1, S)$ on $\mathcal{F}$, such that $\mathcal{L}_1 \mid_\Delta = \mathcal{L}$, and such that $N_{\mathcal{L}_1}(R) = N_{\mathcal{L}}(R)$. The minimality of $|\mathcal{U}|$ yields the existence of a proper locality $(\mathcal{L}^+, \Delta^+, S)$ such that $\mathcal{L}^+ \mid_{\Delta_1} = \mathcal{L}_1$ and with $\mathcal{F}_S(\mathcal{L}^+) = \mathcal{F}$. Then $\mathcal{L}^+ \mid_\Delta = \mathcal{L}$. The explicit construction of $\mathcal{L}^+$ in section 3 shows that every element of $\mathcal{L}^+$ is a Π^+-product of elements of $\mathcal{L}$, so we have (a). Point (b) is then immediate from the corresponding uniqueness result for $\mathcal{L}_1$ (given by Theorem 3.3) and for $\mathcal{L}^+$ with respect to $\mathcal{L}_1$. $\square$

5.3 (Theorem A2). *Let the hypothesis and notation be as in Theorem A1. Let $\mathfrak{N}$ be the set of partial normal subgroups of $\mathcal{L}$, and let $\mathfrak{N}^+$ be the set of partial normal subgroups of $\mathcal{L}^+$. For each $\mathcal{N} \in \mathfrak{N}$ let $\mathcal{N}^{\mathcal{L}^+}$ be the set of all elements of $\mathcal{L}^+$ of the form $\Pi^+(g^{-1}, f, g)$, such that $f \in \mathcal{N}$ and such that $(g^{-1}, f, g) \in \mathbf{D}^+$. Then there is a bijection*

$$\mathfrak{N} \to \mathfrak{N}^+ \quad (\mathcal{N} \mapsto \langle \mathcal{N}^{\mathcal{L}^+} \rangle)$$

where $\langle \mathcal{N}^{\mathcal{L}^+} \rangle$ is the partial subgroup of $\mathcal{L}^+$ generated by $\mathcal{N}^{\mathcal{L}^+}$. The inverse of this bijection is the mapping

$$\mathfrak{N}^+ \to \mathfrak{N} \quad (\mathcal{N}^+ \mapsto \mathcal{N}^+ \cap \mathcal{L}).$$

In particular, we have $S \cap \langle \mathcal{N}^{\mathcal{L}^+} \rangle = S \cap \mathcal{N}$ for each $\mathcal{N} \in \mathfrak{N}$.

Proof. As in the proof of Theorem A1, assume that $\mathcal{L}$ is a counter-example with $|\Delta^+ - \Delta|$ as small as possible. Define R, Δ_1, and $\mathcal{L}_1$ as in the preceding proof. Let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup, let $\mathcal{N}_1$ be the partial subgroup of $\mathcal{L}_1$ generated by the set $\mathcal{N}^{\mathcal{L}_1}$ of $\mathcal{L}_1$-conjugates of elements of $\mathcal{N}$, and let $\mathcal{N}^+$ be the partial subgroup of $\mathcal{L}^+$ generated by the set $(\mathcal{N}_1)^{\mathcal{L}^+}$ of $\mathcal{L}^+$-conjugates of elements of $\mathcal{N}_1$. Then Theorem 4.1 yields $\mathcal{N}_1 \trianglelefteq \mathcal{L}$ and $\mathcal{N}_1 \cap \mathcal{L} = \mathcal{N}$, while the minimality of $|\Delta^+ - \Delta|$ yields $\mathcal{N}^+ \trianglelefteq \mathcal{L}^+$ and $\mathcal{N}^+ \cap \mathcal{L}_1 = \mathcal{N}_1$. Then

$$\mathcal{N}^+ \cap \mathcal{L} = \mathcal{N}^+ \cap \mathcal{L}_1 \cap \mathcal{L} = \mathcal{N}_1 \cap \mathcal{L} = \mathcal{N}.$$

Thus, it only remains to show that $\mathcal{N}^+ = \langle \mathcal{N}^{\mathcal{L}^+} \rangle$ as partial subgroups of $\mathcal{L}^+$. For this, the partial normality of $\mathcal{N}^+$ in $\mathcal{L}^+$ implies that it is enough to show that $\mathcal{N}_1 \subseteq \langle \mathcal{N}^{\mathcal{L}^+} \rangle$.

Set $Y_0 = \mathcal{N}^{\mathcal{L}_1}$, and recursively define subsets Y_k of $\mathcal{L}_1$, for $k > 0$, by

$$Y_k = \{\Pi_1(w) \mid w \in \mathbf{W}(\mathcal{Y}_{k-1}) \cap \mathbf{D}_1\},$$

where $\Pi_1 : \mathbf{D}_1 \to \mathcal{L}_1$ is the product in $\mathcal{L}_1$. Then $\mathcal{N}_1$ is the union of the sets Y_k for $k \geq 0$, by I.1.9.

Let $\Pi^+ : \mathbf{D}^+ \to \mathcal{L}^+$ be the product in $\mathcal{L}^+$. Suppose now by way of contradiction that $\mathcal{N}_1 \nsubseteq \langle \mathcal{N}^{\mathcal{L}^+} \rangle$, and let k be the least index for which there exists $f \in Y_k$ and $g \in \mathcal{L}^+$ such that $(g^{-1}, f, g) \in \mathbf{D}^+$ and $\Pi^+(g^{-1}, f, g) \notin \mathcal{N}^+$. Then $f \notin \mathcal{N}$, so $f \in \mathcal{L}_1 - \mathcal{L}$. Setting $U = S_f$, we conclude that $U \in R^{\mathcal{F}}$.

Suppose first that $k = 0$. Then there exists $x \in \mathcal{N}$ and $h \in \mathcal{L}_1$ with $(h^{-1}, x, h) \in \mathbf{D}_1$ and with $f = \Pi_1(h^{-1}, x, h)$. Then $U = S_{(h^{-1},x,h)}$. Setting $v = (g^{-1}, h^{-1}, x, h, g)$, we obtain $S_v = S_{(g^{-1},f,g)} \in \Delta^+$, and then $f^g = x^{hg}$ (as conjugations performed in $\mathcal{L}^+$). Thus $(Y_0)^{\mathcal{L}^+} \subseteq \mathcal{N}^{\mathcal{L}^+}$, and so $k > 0$.

Let $w = (f_1, \cdots, f_n) \in \mathbf{W}(Y_{k-1}) \cap \mathbf{D}_1$ with $f = \Pi_1(w)$. Again, we find that $f \notin \mathcal{L}$, and $S_f = S_w \in R^{\mathcal{F}}$. Set

$$w' = (g^{-1}, f_1, g, \cdots, g^{-1}, f_n, g) \quad \text{and} \quad w'' = (g^{-1}) \circ w \circ (g).$$

Then

$$S_{w'} = S_{w''} = S_{(g^{-1},f,g)} = (S_f \cap S_g)^g.$$

As $(g^{-1}, f, g) \in \mathbf{D}^+$ we obtain $f^g = \Pi^+(f_1^g, \cdots, f_n^g)$. Here $(f_i)^g \in \langle \mathcal{N}^{\mathcal{L}^+} \rangle$ for all i, by the minimality of k, and thus $f^g \in \langle \mathcal{N}^{\mathcal{L}^+} \rangle$. This completes the proof that $\mathcal{N}^+ = \langle \mathcal{N}^{\mathcal{L}^+} \rangle$, and thereby completes the proof of Theorem A2. $\square$

Here is an application. Note that by [He1] the product of partial normal subgroups of a locality is again a partial normal subgroup.

Corollary 5.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be partial normal subgroups of the proper locality $(\mathcal{L}, \Delta, S)$, and let $(\mathcal{L}^+, \Delta^+, S)$ be a proper expansion of $\mathcal{L}$. Then $(\mathcal{M}\mathcal{N})^+ = \mathcal{M}^+\mathcal{N}^+$.*

Proof. For any pair X and Y of non-empty subsets of $\mathcal{L}$ it follows from I.1.9 that $\langle X\rangle\langle Y\rangle$ is a subset of $\langle XY\rangle$. Then $\langle \mathcal{M}^{\mathcal{L}^+} > \langle \mathcal{N}^{\mathcal{L}^+}\rangle \subseteq \langle \mathcal{M}^{\mathcal{L}^+}\mathcal{N}^{\mathcal{L}^+}\rangle$, and so $\mathcal{M}^+\mathcal{N}^+ \leq (\mathcal{M}\mathcal{N})^+$. The reverse inclusion follows from the observation that $\mathcal{M}\mathcal{N} \leq \mathcal{L} \cap \mathcal{M}^+\mathcal{N}^+$. $\square$

We wish also to obtain a version of Theorem A for localities which are homomorphic images of proper localities. Thus, for the remainder of this section $(\mathcal{L}, \Delta, S)$ is a locality (not necessarily proper) on $\mathcal{F}$, and $\mathcal{N} \trianglelefteq \mathcal{L}$ is a partial normal subgroup of $\mathcal{L}$. Set $T = S \cap \mathcal{N}$.

For any $g \in \mathcal{L}$, $\mathcal{N}g$ denotes the set of all products xg such that $x \in \mathcal{N}$ and $(x, g) \in \mathbf{D}$, and we say that $\mathcal{N}g$ is a right coset of $\mathcal{N}$ in $\mathcal{L}$. The analogous notion of left coset is obvious. The set of all cosets (left or right) of $\mathcal{N}$ in $\mathcal{L}$ is partially ordered by inclusion, and one thus has the notion of the maximal cosets of $\mathcal{N}$ in $\mathcal{L}$.

By I.3.14 the maximal left cosets of $\mathcal{N}$ are the maximal right cosets, and these maximal cosets form a partition $\overline{\mathcal{L}} = \mathcal{L}/\mathcal{N}$ of $\mathcal{L}$. Let $\rho : \mathcal{L} \to \overline{\mathcal{L}}$ be the map which sends $g \in \mathcal{L}$ to the unique maximal coset of $\mathcal{N}$ containing g. By I.3.16 there is a unique partial group structure on $\overline{\mathcal{L}}$ which makes ρ into a homomorphism of partial groups, and then the induced map

$$\rho^* : \mathbf{D}(\mathcal{L}) \to \mathbf{D}(\overline{\mathcal{L}})$$

is surjective. A homomorphism from a locality to a partial group is by definition a *projection* (cf. I.4.4) if the induced map between the domains of their products is surjective. Thus, ρ is a projection, and it is shown in I.4.3 that $\overline{\mathcal{L}}$ thereby has the structure of a locality

$$(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S}),$$

where $\overline{D} = \{P\rho \mid P \in \Delta\}$, and where $\overline{S} = S\rho \cong S/T$. An important (and elementary) property of any homomorphism of partial groups (I.1.15) is that it sends subgroups to subgroups.

The following lemma clarifies the relationship between the fusion system $\mathcal{F}$ of $\mathcal{L}$ and the fusion system of a homomorphic image of $\mathcal{L}$.

Lemma 5.4. *Let $(\mathcal{L}, \Delta, S)$ be a locality on $\mathcal{F}$, and let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup. Let $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ be the corresponding quotient locality, and let $\rho : \mathcal{L} \to \overline{\mathcal{L}}$ be the canonical projection. Set $T = S \cap \mathcal{N}$, and set $\overline{\mathcal{F}} := \mathcal{F}_{\overline{S}}(\overline{\mathcal{L}})$.*

(a) *ρ restricts to a surjective homomorphism $\sigma : S \to \overline{S}$, and σ is fusion-preserving relative to $\mathcal{F}$ and $\overline{\mathcal{F}}$. That is, σ is a homomorphism $\mathcal{F} \to \overline{\mathcal{F}}$ of fusion systems. Moreover, if X and Y are subgroups of S containing T then the induced map*

$$\sigma_{X,Y} : Hom_{\mathcal{F}}(X, Y) \to Hom_{\overline{\mathcal{F}}}(\overline{X}, \overline{Y})$$

is surjective.

(b) *Let $X \leq S$ be a subgroup of S containing T. Then X is fully normalized in $\mathcal{F}$ if and only if $\overline{X}$ is fully normalized in $\overline{\mathcal{F}}$.*

(c) *Let $\overline{X} \in \overline{\mathcal{F}}^c$, and let X be the preimage of $\overline{X}$ in S. Then $X \in \mathcal{F}^c$.*

(d) *Let $\overline{X} \in \overline{\mathcal{F}}^{cr}$, and let X be the preimage of $\overline{X}$ in S. Then $X \in \mathcal{F}^{cr}$.*

(e) *If $\mathcal{F}^{cr} \subseteq \Delta$ then $\overline{\mathcal{F}}^{cr} \subseteq \overline{\Delta}$.*

Proof. As ρ is a projection we have $\overline{\Delta} = \{P\rho \mid P \in \Delta\}$ by definition I.4.4. Thus $\sigma : S \to \overline{S}$ is a surjective homomorphism of groups. For each $w \in \mathbf{W}(\mathcal{L})$ and each subgroup X of S_w, write X^w for the image of U under the composite c_w of the conjugation maps given sequentially by the entries of w. Similarly define $\overline{X}^{\overline{w}}$ when $\overline{w} \in \mathbf{W}(\overline{\mathcal{L}})$ and $\overline{X} \leq \overline{S}_{\overline{w}}$. By definition, $Hom(\mathcal{F})$ is the set of all mappings $c_w : X \to Y$ with $X^w \leq Y \leq S$, and similarly for $Hom(\overline{\mathcal{F}})$. If $w = (g_1, \cdots, g_n)$ there is then a commutative diagram of homomorphisms:

$$\begin{array}{ccc} X & \xrightarrow{c_w} & Y \\ \sigma \downarrow & & \downarrow \sigma \\ \overline{X} & \xrightarrow[c_{w\rho^*}]{} & \overline{Y} \end{array}$$

and thus σ is fusion-preserving. Suppose now that X and Y contain T, and let $\overline{w} \in \mathbf{W}(\overline{\mathcal{L}})$ with $\overline{X}^{\overline{w}} \leq \overline{Y}$. By I.3.11 there exists a ρ^*-preimage w of $\overline{w}$ with $w \in \mathbf{W}(N_{\mathcal{L}}(T))$. Then $X^w \leq Y$, and $\sigma \mid_X \circ c_{\overline{w}} = c_w \circ \sigma \mid Y$ as maps from X to $\overline{Y}$. That is, the $\overline{\mathcal{F}}$-homomorphism $c_{\overline{w}} : \overline{X} \to \overline{Y}$ is in the image of $\sigma_{X,Y}$, and thus (a) holds.

Point (b) is given by the observation that if X is a subgroup of S containing T then $\overline{N_S(X)} = N_{\overline{S}}(\overline{X})$. Now let $\overline{X} \in \overline{\mathcal{F}}^c$, let X be the preimage of $\overline{X}$ in S, and let Y be an $\mathcal{F}$-conjugate of X. Then $\overline{Y}$ is an $\overline{\mathcal{F}}$-conjugate of $\overline{X}$, so $C_{\overline{S}}(\overline{Y}) \leq \overline{Y}$, and hence $C_S(Y) \leq Y$. This proves (c). Now assume further that $\overline{X} \in \overline{\mathcal{F}}^{cr}$, and let $Y \in X^{\mathcal{F}}$ with Y fully normalized in $\mathcal{F}$. Then $\overline{Y}$ is fully normalized in $\overline{\mathcal{F}}$ by (b). Set $Q = O_p(N_{\mathcal{F}}(Y))$. Then $\overline{Q} \leq O_p(N_{\overline{\mathcal{F}}}(\overline{Y}))$ by (a), so $\overline{Q} = \overline{Y}$, and hence $Q = Y$. This shows that $X \in \mathcal{F}^{cr}$ (as defined in 1.1), and proves (d). Point (e) is immediate from (d). $\square$

If $(\mathcal{L}, \Delta, S)$ is a proper locality on $\mathcal{F}$, then the quotient locality $\mathcal{L}/\mathcal{N}$ (see I.4.5) need not be proper. For example, let $(\mathcal{N}, \Gamma, T)$ be a direct product of pair-wise isomorphic proper localities $(\mathcal{N}_i, \Gamma_i, T_i)$ with $1 \leq i \leq k$. This means (cf. I.1.17) that the underlying set of $\mathcal{N}$ is the direct product of the sets $\mathcal{N}_i$, $\mathbf{D}(\mathcal{N})$ is the direct product of the domains $\mathbf{D}(\mathcal{N}_i)$ (with the obvious product and inversion), Γ is the direct product of the collections Γ_i, and T is the direct product of the groups T_i. Then, let $\mathcal{L}$ be the partial group obtained as the natural semi-direct product $\mathcal{N} \rtimes H$, where H is the symmetric group on k letters. (The reader should have no difficulty, at this stage, in defining the partial group $\mathcal{L}$.) Let S be a maximal p-subgroup of $\mathcal{L}$ containing T, and let Δ be the set of all $P \leq S$ such that $P \cap T \in \Gamma$. Then $(\mathcal{L}, \Delta, S)$ is a proper locality, while $\mathcal{L}/\mathcal{N}$ is isomorphic to the group H, which is a proper locality if and only if $k = 1$, or (k,p) is $(2,2)$, $(3,3)$, or $(4,2)$. Thus, Theorems A1 and A2 are not directly applicable to homomorphic images of proper localities.

Theorem 5.5. *Let $(\mathcal{L}, \Delta, S)$ be a locality on $\mathcal{F}$, let $\mathcal{N} \trianglelefteq \mathcal{L}$, let $\overline{\mathcal{L}} = \mathcal{L}/\mathcal{N}$ be the quotient locality, and let $\rho : \mathcal{L} \to \overline{\mathcal{L}}$ be the canonical projection. Assume that $\mathcal{L}$ is proper, and let Δ^+ be an $\mathcal{F}$-closed set of subgroups of S such that $\Delta \subseteq \Delta^+ \subseteq \mathcal{F}^s$. Let $(\mathcal{L}^+, \Delta^+, S)$ be the expansion of $\mathcal{L}$ to Δ^+, set $\overline{\Delta}^+ = \{P\rho \mid P \in \Delta^+\}$, and set $\overline{\mathcal{F}} = \mathcal{F}_{\overline{S}}(\overline{\mathcal{L}})$.*

(a) *There is a locality $(\overline{\mathcal{L}}^+, \overline{\Delta}^+, \overline{S})$ on $\overline{\mathcal{F}}$ whose restriction to $\overline{\Delta}$ is $\overline{\mathcal{L}}$, and $\overline{\mathcal{L}}^+$ is*

unique up to a unique isomorphism which restricts to the identity map on $\overline{\mathcal{L}}$.

(b) *There is a unique projection $\rho^+ : \mathcal{L}^+ \to \overline{\mathcal{L}}^+$ whose restriction to $\mathcal{L}$ is ρ. Moreover, $Ker(\rho^+) = \mathcal{N}^+$, where $\mathcal{N}^+$ is the normal closure of $\mathcal{N}$ in $\mathcal{L}^+$ given by Theorem A2.*

Proof. Note first of all that since $\mathcal{F}^{cr} \subseteq \Delta$ we have $\overline{\mathcal{F}}^{cr} \subseteq \overline{\Delta}$ by 5.4(e). Notice also that if $\Delta = \Delta^+$ then there is nothing to prove, so we may assume that Δ is properly contained in Δ^+.

Among all $\mathcal{F}$-closed sets Δ_1 with $\Delta \subseteq \Delta_1 \subseteq \Delta^+$, let Δ_1 be maximal subject to the condition that (a) and (b) hold with Δ_1 in the role of Δ^+. Then Δ_1 is a proper subset of Δ^+, and by replacing Δ with Δ_1 we reduce (as in the proof of Theorem A1) to the case where $\Delta^+ = \Delta \cup R^{\mathcal{F}}$ for some $R \leq S$.

Take R to be fully normalized in $\mathcal{F}$, and suppose that $T \nleq R$. Then $RT \in \Delta$, and then $\Delta^+\rho = \Delta\rho = \overline{\Delta}$. Then $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is the unique locality on $\overline{\mathcal{F}}$ whose restriction to $\overline{\Delta}$ (namely, itself) is $\overline{\mathcal{L}}$, and thus (a) holds in this case. In order to verify (b) in this case recall that, by Theorem A1, $\mathcal{L}^+$ is the "free amalgamated product" in the category of partial groups of the "amalgam" given by the inclusion maps $\mathcal{L}_0 \to \mathcal{L}$ and $\mathcal{L}_0 \to \mathcal{L}_0^+$. Point (b) will follow if:

(1) There is a unique homomorphism $\rho_0^+ : \mathcal{L}_0^+ \to \overline{\mathcal{L}}$ such that ρ_0^+ agrees with ρ on $\mathcal{L}_0$.

(2) ρ_0+ induces a surjection $\mathbf{D}(\mathcal{L}_0^+) \to \mathbf{D}(\overline{\mathcal{L}}_0)$, where $\overline{\mathcal{L}}_0$ is the set of all $\overline{f} \in \overline{\mathcal{L}}$ such that $\overline{S}_{\overline{f}}$ contains a $\overline{\mathcal{F}}$-conjugate of $\overline{R}$.

(3) $Ker(\rho^+) = \mathcal{N}^+$.

Versions of the same three points will need to be verified in the case where $T \leq R$, and our approach will be to merely sketch the proofs in each case, leaving some of the entirely mechanical details to the reader. Thus, let Φ be the set of triples (x^{-1}, g, y) defined following 3.4, let $\approx$ be the relation on Φ defined in 3.10, and let $\rho^* : \mathbf{W}(\mathcal{L}) \to \mathbf{W}(\overline{\mathcal{L}})$ be the ρ-induced homomorphism of free monoids. Then ρ^* maps Φ into $\mathbf{D}(\overline{\mathcal{L}})$, and it may be routinely verified that the composition

$$\Phi \xrightarrow{\rho^*} \mathbf{D}(\overline{\mathcal{L}}) \xrightarrow{\overline{\Pi}} \overline{\mathcal{L}}$$

is constant on $\approx$-classes. Thus, there is a mapping $\rho_0^+ : \mathcal{L}_0^+ \to \overline{\mathcal{L}}$ given by $[x^{-1}, g, y] \mapsto \overline{\Pi}(\overline{x}^{-1}, \overline{g}, \overline{y})$. The verification that ρ_0^+ is a homomorphism is also routine, and yields (1). Since ρ^* maps $\mathbf{D}(\mathcal{L})$ onto $\mathbf{D}(\overline{\mathcal{L}})$, (2) is then immediate. As $Ker(\rho^+) \cap \mathcal{L} = \mathcal{N}$, (3) is immediate from Theorem A2. Thus, the theorem holds if $T \nleq R$.

Assume henceforth that $T \leq R$. Then $N_{\overline{\mathcal{L}}}(\overline{R}) = \overline{N_{\mathcal{L}}(R))}$, since each element of $\overline{\mathcal{L}}$ has a ρ-preimage in $N_{\mathcal{L}}(T)$ by I.3.11. Then $N_{\overline{\mathcal{L}}}(\overline{R})$ is a subgroup of $\overline{\mathcal{L}}$ by I.1.15. Thus Hypothesis 3.2 is satisfied, with $\overline{\mathcal{L}}$ and $\overline{R}$ in the roles of $\mathcal{L}$ and R. Theorem 3.3 then yields (a). For the same reasons as in the preceding case (and because the verification of (3) did not in fact make use of the hypothesis that $T \nleq R$) it now suffices to verify:

(4) There is a unique homomorphism $\rho_0^+ : \mathcal{L}_0^+ \to \overline{\mathcal{L}}^+$ such that ρ_0^+ agrees with ρ on $\mathcal{L}_0$.

(5) ρ_0+ induces a surjection $\mathbf{D}(\mathcal{L}_0^+) \to \mathbf{D}(\overline{\mathcal{L}_0^+})$, where $\overline{\mathcal{L}_0^+}$ is the set of all $\overline{f} \in \overline{\mathcal{L}}^+$ such that $\overline{S}_{\overline{f}}$ contains a $\overline{\mathcal{F}}$-conjugate of $\overline{R}$.

Let Φ and ρ^* be as above, and define $\overline{\Phi}$ to be the set of triples $\overline{\phi} = (\overline{x}^{-1}, \overline{g}, \overline{y})$ such that one has

$$\overline{U} \xrightarrow{\overline{x}^{-1}} \overline{R} \xrightarrow{\overline{g}} \overline{R} \xrightarrow{\overline{y}} \overline{V}$$

(a sequence of conjugation isomorphisms between subgroups of $\overline{S}$, labeled by the conjugating elements), with $N_{\overline{S}}(\overline{U}) \leq \overline{S}_{\overline{x}^{-1}}$, and with $N_{\overline{S}}(\overline{V}) \leq \overline{S}_{\overline{y}^{-1}}$. Then ρ^* maps Φ onto $\overline{\Phi}$ by 5.4(a). It need cause no confusion to denote also by $\sim$ and $\approx$ the two equivalence relations on $\overline{\Phi}$ given by direct analogy with 3.4 and 3.9. Again by means of 5.4(a), one verifies that the restriction of ρ^* to Φ preserves these equivalence relations, and hence induces a surjective mapping $\rho_0^+ \mathcal{L}_0^+ \to \overline{\mathcal{L}}_0^+$. Here $\mathcal{L}_0$ is by definition the set of elements $f \in \mathcal{L}$ such that S_f contains an $\mathcal{F}$-conjugate of R, and any such f is identified with its $\approx$-class $[f]$, consisting of all those $\phi \in \Phi \cap \mathbf{D}(\mathcal{L})$ such that $\Pi(\phi) = f$. The analogous definition of $\overline{\mathcal{L}}_0$ leads to the conclusion that the restriction of ρ_0^+ to $\mathcal{L}_0$ is the (surjective) homorphism $\rho_0 : \mathcal{L}_0 \to \overline{\mathcal{L}}_0$ induced by ρ. The verification that ρ_0^+ is a homomorphism, and hence that (4) holds, is then straightforward.

Set $\mathbf{D}_0^+ = \mathbf{D}(\mathcal{L}_0^+)$ and $\overline{\mathbf{D}}_0^+ = \mathbf{D}(\overline{\mathcal{L}_0}^+)$, let $\overline{w} \in \overline{\mathbf{D}}_0^+$, and write $\overline{w} = ([\overline{\phi}_1], \cdots, [\overline{\phi}_n])$. Here $[\overline{\phi}_i]$ is the $\approx$-class of an element $\overline{\phi}_i = (\overline{x}_i^{-1}, \overline{g}_i, \overline{y}_i)$ in $\overline{\Phi}$, and the representatives $\overline{\phi}_i$ may be chosen so that the sequence $\overline{\gamma} = (\overline{\phi}_1, \cdots, \overline{\phi}_n)$ is a $\overline{\Gamma}$-form of $\overline{w}$, as defined in 3.11. That is, the word

$$\overline{w}_{\overline{g}} = \overline{\phi}_1 \circ \cdots \circ \overline{\phi}_n$$

has the property that $\overline{S}_{\overline{w}_{\overline{\gamma}}}$ contains a $\overline{\mathcal{F}}$-conjugate of $\overline{R}$. Let ϕ_i be a ρ^*-preimage of $\overline{\phi}_i$ in Φ, set $\gamma = (\phi_1, \cdots, \phi_n)$, let $[\phi_i]$ be the $\approx$-class of ϕ_i (in $\mathcal{L}_0^+$), and set $w = ([\phi_1], \cdots, [\phi_n])$. One verifies that γ is a Γ-form of w, and hence that $w \in \mathbf{D}_0^+$. Since ρ^* maps w_γ to $\overline{w}_{\overline{\gamma}}$, it follows that ρ_0^+ induces a surjection as required in (5). Thus (5) holds, and the proof is complete. $\square$

The preceding result is essentially a generalization of Theorem A1 to homomorphic images of proper localities. Here is the corresponding version of Theorem A2.

Theorem 5.6. *Let the hypotheses and the setup be as in the preceding theorem. Let $\overline{\mathcal{K}}$ be a partial normal subgroup of $\overline{\mathcal{L}}$, and let $\mathcal{K}$ be the ρ-preimage of $\overline{\mathcal{K}}$ in $\mathcal{L}$ (a partial normal subgroup of $\mathcal{L}$, by I.4.7). Let $\mathcal{K}^+ = \langle \mathcal{K}^{\mathcal{L}^+} \rangle$ be the partial normal subgroup of $\mathcal{L}^+$ given by Theorem A2, and let $\overline{\mathcal{K}}^+ = \langle \overline{\mathcal{K}}^{\overline{\mathcal{L}}^+} \rangle$ be the partial subgroup of $\overline{\mathcal{L}}^+$ generated by the set of $\overline{\mathcal{L}}^+$-conjugates of elements of $\overline{\mathcal{K}}$. Then:*

(a) *ρ^+ restricts to a surjective homomorphism $\mathcal{K}^+ \to \overline{\mathcal{K}}^+$ of partial groups, with kernel $\mathcal{N}^+$. In particular, $\overline{\mathcal{K}^+} = \overline{\mathcal{K}}^+$.*

(b) *$\overline{\mathcal{K}}^+ \trianglelefteq \mathcal{L}^+$, and $\overline{\mathcal{K}}^+ \cap \overline{\mathcal{L}} = \overline{\mathcal{K}}$.*

Proof. Let $\mathfrak{K}$ be the set of partial normal subgroups of $\mathcal{L}$ containing $\mathcal{N}$, $\mathfrak{K}^+$ the set of partial normal subgroups of $\mathcal{L}^+$ containing $\mathcal{N}^+$, $\overline{\mathfrak{K}}$ the set of partial normal subgroups of

$\overline{\mathcal{L}}$, and $\overline{\mathfrak{K}}^+$ the set of partial normal subgroups of $\overline{\mathcal{L}}^+$. By I.4.7 there are bijections

$$\mathfrak{K} \to \overline{\mathfrak{K}} \ \text{ and } \ \mathfrak{K}^+ \to \overline{\mathfrak{K}}^+,$$

by which a partial normal subgroup is sent to its image in $\overline{\mathcal{L}}$ (or in $\overline{\mathcal{L}}^+$) under the canonical projection ρ (or ρ^+). By Theorem A2 there is a bijection $\mathfrak{K} \to \mathfrak{K}^+$ which sends $\mathcal{K} \in \mathfrak{K}$ to $\mathcal{K}^+ = \langle \mathcal{K}^{\mathcal{L}^+} \rangle$. These three bijections form three sides of an obvious commutative diagram whose fourth side is a bijection

$$(*) \qquad \overline{\mathfrak{K}} \to \overline{\mathfrak{K}}^+ \ \ (\overline{\mathcal{K}} \mapsto \overline{\mathcal{K}}^+)$$

As ρ^+ maps $\mathbf{D}(\mathcal{L}^+)$ onto $\mathbf{D}(\overline{\mathcal{L}}^+)$, the correspondence (*) is given by

$$\overline{\mathcal{K}}^+ = \langle \overline{\mathcal{K}}^{\overline{\mathcal{L}}^+} \rangle,$$

which yields the theorem. $\square$

Section 6: The fusion system of a proper locality

We have avoided the notion of "saturation" for fusion systems so far in this work, and in fact we are seeking a substitute for it which may be more in harmony with the approach via localities. Theorem 6.1 will provide such a substitute; but in the interest of giving a short proof we shall in fact rely on known results in [AKO], [5a], and [He2] on saturated fusion systems. (The situation will be different in the parallel series of papers with Gonzalez where, other than in some special cases, there will be no such results on which to rely.) The reader should recall the definition of inductive fusion system from 1.11. For the definitions of saturated fusion system on S, fully automized subgroup of S, and receptive subgroup of S, we refer the reader to [definition I.2.2 in AKO].

Theorem 6.1. *Let $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$ be the fusion system of a proper locality $(\mathcal{L}, \Delta, S)$. Then $\mathcal{F}$ is inductive; and for each $V \leq S$ with V fully normalized in $\mathcal{F}$ the fusion systems $N_{\mathcal{F}}(V)$ and $C_{\mathcal{F}}(V)$ are (cr)-generated.*

Proof. Let $Q \in \Delta$. Then each $\mathcal{F}$-automorphism of Q is given by conjugation by an element of $N_{\mathcal{L}}(Q)$, and so $Aut_{\mathcal{F}}(Q) \cong N_{\mathcal{L}}(Q)/C_{\mathcal{L}}(Q)$. Assume that Q is fully normalized in $\mathcal{F}$. Then $N_S(Q) \in Syl_p(N_{\mathcal{L}}(Q)$ by 2.1, and hence $Aut_S(Q) \in Syl_p(Aut_{\mathcal{F}}(Q))$. That is, Q is fully automized in $\mathcal{F}$. Now let P be an $\mathcal{F}$-conjugate of Q, let $\phi : P \to Q$ be an $\mathcal{F}$-isomorphism, and let R be a subgroup of $N_S(P)$ such that

$$(*) \qquad \phi^{-1} Aut_R(P) \phi \leq Aut_S(Q).$$

Then ϕ is conjugation by an element $g \in \mathcal{L}$, and I.2.3(b) shows that g-conjugation is an isomorphism from $N_{\mathcal{L}}(P)$ to $N_{\mathcal{L}}(Q)$. Thus R^g is a p-subgroup of $N_{\mathcal{L}}(Q)$, and the condition (*) implies that $R^g \leq C_{\mathcal{F}}(Q) N_S(Q)$. As $N_S(Q) \in Syl_p(C_{\mathcal{F}}(Q) N_S(Q))$ there

exists $h \in C_{\mathcal{F}}(Q)$ such that $R^{gh} \leq N_S(Q)$. Then conjugation by gh is an extension of ϕ to an $\mathcal{F}$-homomorphism $\overline{\phi} : R \to N_S(Q)$; which is to say that Q is receptive in $\mathcal{F}$. Thus $\mathcal{F}$ is Δ-saturated. By Theorem A1 we may assume $\mathcal{F}^c \subseteq \Delta$, and then [Theorem 2.2 in 5a] implies that $\mathcal{F}$ is saturated.

Let $Y \leq S$ be fully normalized in $\mathcal{F}$, and let X be an $\mathcal{F}$-conjugate of Y. Any $\mathcal{F}$-isomorphism $X \to Y$ conjugates $Aut_{\mathcal{F}}(X)$ to $Aut_{\mathcal{F}}(Y)$. As $\mathcal{F}$ is saturated, Y is fully automized, and so there exists an $\mathcal{F}$-isomorphism $\psi : X \to Y$ such that ψ conjugates $Aut_S(X)$ into $Aut_S(Y)$. As also Y is receptive, ψ extends to an $\mathcal{F}$-homomorphism $N_S(X) \to N_S(Y)$, and thus $\mathcal{F}$ is inductive. By [Theorem I.5.5 in AKO], $N_{\mathcal{L}}(X)$ and $C_{\mathcal{L}}(X)$ are saturated, and these fusion systems are then (cr)-generated as an immediate consequence of Alperin's fusion theorem [Theorem I.3.5 in AKO]. $\square$

We wish to expand on 6.1, by showing that if V is fully normalized in $\mathcal{F}$ then $N_{\mathcal{F}}(V)$ and $C_{\mathcal{F}}(V)$ are fusion systems of proper localities. The following lemma will be needed.

Lemma 6.2. *Let $\mathcal{F}$ be the fusion system of a proper locality, let V be fully normalized in $\mathcal{F}$, and let P be a subgroup of $N_S(V)$ containing V. Then P is centric in $\mathcal{F}$ if and only if P is centric in $N_{\mathcal{F}}(V)$.*

Proof. Set $\mathcal{F}_V = N_{\mathcal{F}}(V)$. We are free to replace P by any $\mathcal{F}_V$-conjugate of P, so we may assume that P is fully normalized in $\mathcal{F}_V$. As $\mathcal{F}$ is inductive by 6.1, P is then fully centralized in $\mathcal{F}$ by 1.16. Then $P \in \mathcal{F}^c$ if and only if $C_S(P) \leq P$ by 1.10. As $V \leq P$ we have $C_S(P) = C_{N_S(V)}(P)$, and the lemma follows. $\square$

Proposition 6.3. *Let $(\mathcal{L}, \Delta, S)$ be a proper locality on $\mathcal{F}$, and let $V \leq S$ be a subgroup of S such that V is fully normalized in $\mathcal{F}$. Then there exists a proper locality $(\mathcal{L}_V, \Delta_V, N_S(V))$ on $N_{\mathcal{F}}(V)$, and a proper locality $(\mathcal{C}_V, \Sigma_V, N_S(V))$ on $C_{\mathcal{F}}(V)$.*

Proof. By Theorem A1 we may take $\Delta = \mathcal{F}^c$. Set $\mathcal{F}_V = N_{\mathcal{F}}(V)$, and set $\Delta_V = (\mathcal{F}_V)^c$. Further, set

$$\mathcal{L}_V = \{g \in N_{\mathcal{L}}(V) \mid N_{S_g}(V) \in \Delta_V\},$$

and write

$$\mathbf{D}_V = \mathbf{D}_{\Delta_V} = \{w \in \mathbf{W}(N_{\mathcal{L}}(V)) \mid N_{S_w}(V) \in \Delta_V\}.$$

Then 6.2 yields $\mathcal{D}_V \subseteq \mathcal{F}^c$, and $\Delta_V = \{P \in \Delta \mid V \trianglelefteq P\}$. Notice that $\Pi(w) \in \mathcal{L}_V$ for any $w \in \mathbf{D}_V$, by I.2.5(c). It is then a straightforward exercise with definition I.1.1 to verify that $\mathcal{L}_V$ is a partial group with respect to the restriction $\Pi_V : \mathbf{D}_V \to \mathcal{L}_V$ of Π, and with respect to the restriction to $\mathcal{L}_V$ of the inversion in $\mathcal{L}$. Since $(\mathcal{F}_V)^c$ is $\mathcal{F}_V$-closed, it is immediate from the above definition of Δ_V and from definition II.2.1 that $(\mathcal{L}_V, \Delta_V)$ is objective. As V is fully normalized in $\mathcal{F}$, $N_S(V)$ is a maximal p-subgroup of $\mathcal{L}_V$ by I.2.11(b). Thus $(\mathcal{L}_V, \Delta_V, N_S(V)$ is a locality. Set $\mathcal{E}_V = \mathcal{F}_{N_S(V)}(N_{\mathcal{L}}(V))$. Then $\mathcal{E}_V$ is a fusion subsystem of $\mathcal{F}_V$. Since $(\mathcal{F}_V)^{cr} \subseteq \Delta_V$, and since $\mathcal{F}_V$ is (cr)-generated by 6.1, it follows that $\mathcal{E}_V = \mathcal{F}_V$. Let $P \in \Delta_V$. Then

$$N_{\mathcal{L}_V}(P) = N_{N_{\mathcal{L}(V)}}(P) = N_{N_{\mathcal{L}(P)}}(V),$$

and hence $N_{\mathcal{L}_V}(P)$ is a group of characteristic p by 2.7(b). Thus $(\mathcal{L}_V, \Delta_V, N_S(V)$ is a proper locality on $N_{\mathcal{F}}(V)$.

We may assume henceforth that $\mathcal{L} = \mathcal{L}_V$. Set $\Sigma = C_{\mathcal{F}}(V)^c$ and define $C_{\mathcal{L}}(V)$ to be the set of all $g \in \mathcal{L}$ such that $g^x = g$ for all $x \in V$. One observes that $\mathcal{C}_{\mathcal{L}}(V)$ is a partial normal subgroup of $\mathcal{L}$, with $S \cap \mathcal{C}_{\mathcal{L}}(V) = C_S(V)$. Set $H = N_{\mathcal{L}}(C_S(V))$. Then $\mathcal{L} = H\mathcal{C}_{\mathcal{L}}(V)$ by the Frattini Lemma (I.3.11), and 1.5 shows that H acts on $C_{\mathcal{F}}(V)$. Then Σ is $\mathcal{F}$-invariant. Now let $X \in \Sigma$. Then each $\mathcal{F}$-conjugate of VX is of the form VY where $Y \in X^{\mathcal{F}}$. Then $Y \in \Sigma$, and so

$$C_S(VY) = C_{C_S(V)}(Y) \leq Y \leq VY.$$

This shows that $VX \in \mathcal{F}^c$ for $X \in \Sigma$.

Set $\mathcal{C}_V = \{g \in C_{\mathcal{L}}(V) \mid C_{S_g}(V) \in \Sigma\}$, and write

$$\mathbf{E} = \mathbf{D}_\Sigma = \{w \in \mathbf{W}(\mathcal{C}_V) \mid C_{S_w}(V) \in \Sigma\}.$$

Then $\mathbf{E} \subseteq \mathbf{D}$, and Π restricts to a mapping $\mathbf{E} \to \mathcal{C}_V$ which, together with the restriction of the inversion in $\mathcal{L}$, makes $\mathcal{C}_V$ into a partial group, and which makes $(\mathcal{C}_V, \Sigma)$ into an objective partial group. As $C_S(V)$ is a maximal p-subgroup of $C_{\mathcal{L}}(V)$, by I.3.1(c), $C_S(V)$ is also a maximal p-subgroup of C_V. Thus $(\mathcal{C}_V, \Sigma, C_S(V))$ is a locality. As $C_{\mathcal{F}}(V)^{cr} \subseteq \Sigma$ we find that $C_{\mathcal{F}}(V) = \mathcal{F}_{C_S(V)}(\mathcal{C}_V)$. For $X \in \Sigma$, $N_{\mathcal{C}_V}(X)$ is a normal subgroup of the group $N_{\mathcal{L}}(VX)$, and then 2.7(a) shows that $N_{\mathcal{C}_V}(X)$ is of characteristic p. Thus $(\mathcal{C}_V, \Sigma, C_S(V))$ is a proper locality on $C_{\mathcal{F}}(V)$. $\square$

The next few results concern the set $\mathcal{F}^s$ of $\mathcal{F}$-subcentric subgroups of the fusion system $\mathcal{F}$ of a proper locality; and the proofs are variations of the proofs of corresponding results in [He2]. There are several reasons for providing a complete treatment here, and for not simply quoting from [He2]. One of these is that all that will be needed from Theorem 6.1 (whose proof mainly consisted in showing that $\mathcal{F}$ is saturated) is that $\mathcal{F}$ be inductive. A second reason is that we wish to have arguments which will apply in a more general setup in which $\mathcal{L}$ is not necessarily finite.

The key insight into $\mathcal{F}^s$ is given by [Lemma 3.1 in He2], which we prove here as follows.

Lemma 6.4. *Let $\mathcal{F}$ be the fusion system on S of a proper locality, and let $V \leq S$ be fully centralized in $\mathcal{F}$. Then $V \in \mathcal{F}^s$ if and only if $O_p(C_{\mathcal{F}}(V))$ is centric in $C_{\mathcal{F}}(V)$.*

Proof. If $U \in V^{\mathcal{F}}$ is fully normalized in $\mathcal{F}$ then 1.14 yields an isomorphism $C_{\mathcal{F}}(U) \cong C_{\mathcal{F}}(V)$ of fusion systems. We may therefore assume to begin with that V is fully normalized in $\mathcal{F}$. Set $\mathcal{F}_V = N_{\mathcal{F}}(V)$ and $R = O_p(\mathcal{F}_V)$. Also, set $\mathcal{C}_V = C_{\mathcal{F}}(V)$ and $Q = O_p(\mathcal{C}_V)$.

Suppose first that the lemma holds with $\mathcal{F}_V$ in place of $\mathcal{F}$. That is, suppose that R is centric in $\mathcal{F}_V$ if and only if Q is centric in $\mathcal{C}_V$. Since $R \in (\mathcal{F}_V)^c$ if and only if $R \in \mathcal{F}^c$ by 6.2, we then have the lemma in general. Thus, we are reduced to the case where $V \trianglelefteq \mathcal{F}$, and $R = O_p(\mathcal{F})$.

Fix a proper locality $(\mathcal{L}, \Delta, S)$ on $\mathcal{F}$. We may assume that $\mathcal{F}^c \subseteq \Delta$ by Theorem A1. Suppose that $V \in \mathcal{F}^s$. Then $R \in \mathcal{F}^c$, so $R \in \Delta$, and $\mathcal{L}$ is the group $N_{\mathcal{L}}(R)$. Notice that

$[R, C_S(VQ)] \le C_R(VQ) \le Q$, and that both R and Q are normal subgroups of $\mathcal{L}$. Then $C_S(VQ) \le R$ by 2.7(c). Then also

$$C_{C_S(V)}(Q) = C_S(VQ) \le C_R(V) \le Q,$$

and thus Q is centric in $C_{\mathcal{F}}(V)$. On the other hand, assuming that Q is centric in $C_{\mathcal{F}}(V)$, we obtain

$$C_S(R) \le C_S(VQ) \le Q \le R,$$

and so $R \in \mathcal{F}^c$, as required. □

Corollary 6.5. *Let $\mathcal{F}$ be a fusion system of a proper locality. Then*

$$\mathcal{F}^{cr} \subseteq \mathcal{F}^c \subseteq \mathcal{F}^q \subseteq \mathcal{F}^s.$$

Proof. We have $\mathcal{F}^{cr} \subseteq \mathcal{F}^c$ by definition. Let $(\mathcal{L}, \Delta, S)$ be a proper locality on $\mathcal{F}$, and $P \le S$ be fully normalized in $\mathcal{F}$. If $P \in \mathcal{F}^c$ then $C_{\mathcal{F}}(P)$ is the trivial fusion system on $Z(P)$, and so $P \in \mathcal{F}^q$. Now suppose instead that we are given $P \in \mathcal{F}^q$. Then $O_p(C_{\mathcal{F}}(Q)) = C_S(Q)$, and then $P \in \mathcal{F}^s$ by 6.4. □

Corollary 6.6. *Let $\mathcal{F}$ be the fusion system on S of a proper locality, and let $V \le S$ with V fully normalized in $\mathcal{F}$. Then*

$$\{Q \in N_{\mathcal{F}}(V)^s \mid V \le Q\} \subseteq \mathcal{F}^s.$$

Proof. One need only observe that $C_{\mathcal{F}}(V) = C_{C_{\mathcal{F}}(V)}(V)$, in order to obtain the desired result from 6.4. □

The next result is [lemma 3.2 in He].

Theorem 6.7. *Let $\mathcal{F}$ be the fusion system of a proper locality. Then $\mathcal{F}^s$ is $\mathcal{F}$-closed.*

Proof. Fix a proper locality $(\mathcal{L}, \Delta, S)$ on $\mathcal{F}$, with $\Delta = \mathcal{F}^c$. Let U, V be subgroups of S. By 1.14, $N_{\mathcal{F}}(U) \cong N_{\mathcal{F}}(V)$ if U and V are $\mathcal{F}$-conjugate subgroups of S, so $\mathcal{F}^s$ is $\mathcal{F}$-invariant. Clearly $S \in \mathcal{F}^s$. Thus, we are reduced to showing that $\mathcal{F}^s$ is closed with respect to overgroups in S.

Among all $V \in \mathcal{F}^s$ such that some overgroup of V in S is not subcentric in $\mathcal{F}$, choose V so that $|V|$ is as large as possible. Then there exists an overgroup P of V in S such that $P \notin \mathcal{F}^s$ and such that V has index p in P. Then $P \le N_S(V)$. Let $U \in V^{\mathcal{F}}$ such that U is fully normalized in $\mathcal{F}$. As $\mathcal{F}$ is inductive by 6.1, there is an $\mathcal{F}$-homomorphism $\phi : N_S(V) \to N_S(U)$ such that $V\phi = U$. Then $P\phi \notin \mathcal{F}^s$ as $\mathcal{F}^s$ is $\mathcal{F}$-invariant, and so we may assume to begin with that V is fully normalized in $\mathcal{F}$. Set $\mathcal{F}_V = N_{\mathcal{F}}(V)$. Replacing P with a suitable $\mathcal{F}_V$-conjugate of P, we may assume that P is fully normalized in $\mathcal{F}_V$.

Suppose that $P \in (\mathcal{F}_V)^s$. As $C_{\mathcal{F}}(P) = C_{\mathcal{F}_V}(P)$ we then obtain $P \in \mathcal{F}^s$ from 6.4. Thus $P \notin (\mathcal{F}_V)^s$, so $\mathcal{F}_V$ is a counterexample, and we may therefore assume that $\mathcal{F}_V = \mathcal{F}$. Then $O_p(\mathcal{F}) \in \mathcal{F}^c$, so $P \in \Delta$, and $\mathcal{L}$ is a group of characteristic p. Then $N_{\mathcal{L}}(P)$ is of characteristic p by 2.7(b), and since P is fully normalized in $\mathcal{F}$ we have also $N_{\mathcal{F}}(P) = \mathcal{F}_{N_S(P)}(N_{\mathcal{L}}(P))$. Then $O_p(N_{\mathcal{F}}(P)) \in N_{\mathcal{F}}(P)^c$, and hence $O_p(N_{\mathcal{F}}(P)) \in \mathcal{F}^c$ by 6.2. Thus $P \in \mathcal{F}^s$, as required. □

The following result is immediate from 6.7 and Theorem A1.

Corollary 6.8. *Let $\mathcal{F}$ be the fusion system of a proper locality. Then there exists a proper locality $(\mathcal{L}, \Delta, S)$ on $\mathcal{F}$ with $\Delta = \mathcal{F}^s$.* $\square$

The following result is [lemma 3.3 in He].

Lemma 6.9. *Let $\mathcal{F}$ be the fusion system on S of a proper locality, and let $P \leq S$ with $O_p(\mathcal{F})P \in \mathcal{F}^s$. Then $P \in \mathcal{F}^s$.*

Proof. Among all counter-examples, choose P so that $|P|$ is as large as possible. Set $Q = O_p(\mathcal{F})$, and let $\phi : P \to P'$ be an $\mathcal{F}$-isomorphism such that P' is fully normalized in $\mathcal{F}$. Then ϕ extends to an $\mathcal{F}$-isomorphism $PQ \to P'Q$, so $P'Q \in \mathcal{F}^s$. Thus we may assume that P is fully normalized in $\mathcal{F}$.

Set $\mathcal{E} = N_{\mathcal{F}}(P)$ and set $D = N_Q(P)$. Then $D \leq O_p(\mathcal{E})$. If $D \leq P$ then $P = PR$, and P is not a counter-example. Thus $D \nleq P$, and $DP \in \mathcal{F}^s$ by the maximality of $|P|$. Since $N_S(P) \leq N_S(PD)$ we may assume that P was chosen so that also PD is fully normalized in $\mathcal{F}$. Set $R = N_S(PD)$. By 6.8 $\mathcal{F}$ is the fusion system of a proper locality $\mathcal{L}$ whose set of objects is $\mathcal{F}^s$. Then $G := N_{\mathcal{L}}(DP)$ is a group of characteristic p, $R \in Syl_p(G)$, and $\mathcal{F}_R(G) = N_{\mathcal{F}}(DP)$. Since each $\mathcal{E}$-homomorphism extends to an $N_{\mathcal{F}}(DP)$-homomorphism, $\mathcal{E}$ is then the fusion system of $N_G(P)$ at $N_S(P)$. As G is of characteristic p, so is $N_G(P)$, by 2.7(b). Thus $O_p(\mathcal{E})$ is centric in $\mathcal{E}$, and hence centric in $\mathcal{F}$ by 6.2. Thus $P \in \mathcal{F}^s$. $\square$

This ends our treatment of $\mathcal{F}^s$. We end the section with the following technical result, which will be needed in Part III (for the proof of III.9.8).

Lemma 6.10. *Let $\mathcal{L}$ be a proper locality on $\mathcal{F}$, let $P \in \mathcal{F}^{cr}$, let $T \leq S$ be strongly closed in $\mathcal{F}$, and let $\mathcal{E}$ be an inductive fusion system on T such that $\mathcal{E}$ is a subsystem of $\mathcal{F}$. Then there exists $Q \in P^{\mathcal{F}}$ with $Q \cap T \in \mathcal{E}^c$.*

Proof. Set $U = P \cap T$ and set $A = N_{C_T(U)}(P)$. Then $[P, A] \leq U$, and thus A centralizes the chain $(P \geq U \geq 1)$ of normal subgroups of the group $N_{\mathcal{L}}(P)$. Then $A \leq O_p(N_{\mathcal{L}}(P))$ by 2.7(c), and then $A \leq P$ by an application of 2.3 to the fusion systen $N_{\mathcal{F}}(P)$. Thus $C_T(U) \leq P$, and so $C_T(U) \leq U$.

Let $V \in U^{\mathcal{F}}$ be fully normalized in $\mathcal{F}$. As $\mathcal{F}$ is inductive by 6.1, there exists an $\mathcal{F}$-homomorphism $\phi : N_S(U) \to N_S(V)$ with $U\phi = V$. Set $Q = P\phi$. Then $Q \in \mathcal{F}^{cr}$, and so $C_T(V) \leq V$ by the result of the preceding paragraph. Let $V' \in V^{\mathcal{E}}$. Then $V' \in V^{\mathcal{F}}$, and so there exists an $\mathcal{F}$-homomorphism $\psi : N_S(V') \to N_S(V)$ with $V' = V\psi$. Then $N_T(V')\psi \leq N_T(V)$ as T strongly closed in $\mathcal{F}$, and thus $|N_T(V')| \leq |N_T(V)|$. This shows that V is fully normalized in $\mathcal{E}$. As $\mathcal{E}$ is inductive, V is fully centralized in $\mathcal{E}$ by 1.13. As $C_T(V) \leq V$, V is then centric in $\mathcal{E}$ by 1.10. $\square$

Section 7: $O^p_{\mathcal{L}}(\mathcal{N})$ and $O^{p'}_{\mathcal{L}}(\mathcal{N})$

Definition 7.1. Let $(\mathcal{L}, \Delta, S)$ be a locality, let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup of $\mathcal{L}$, and set $T = S \cap \mathcal{N}$. Set

$$\mathbb{K} = \{\mathcal{K} \trianglelefteq \mathcal{L} \mid \mathcal{K}T = \mathcal{N}\} \text{ and } \mathbb{K}' = \{\mathcal{K}' \trianglelefteq \mathcal{L} \mid T \leq \mathcal{K}\},$$

and set

$$O^p_{\mathcal{L}}(\mathcal{N}) = \bigcap \mathbb{K} \ \text{ and } \ O^{p'}_{\mathcal{L}}(\mathcal{N}) = \bigcap \mathbb{K}'.$$

Write $O^p(\mathcal{L})$ for $O^p_{\mathcal{L}}(\mathcal{L})$, and $O^{p'}(\mathcal{L})$ for $O^{p'}_{\mathcal{L}}(\mathcal{L})$.

Proposition 7.2. *$O^p_{\mathcal{L}}(\mathcal{N}) \in \mathbb{K}$ and $O^{p'}_{\mathcal{L}}(\mathcal{N}) \in \mathbb{K}'$*

Proof. Let $\mathcal{K}_1, \mathcal{K}_2 \in \mathbb{K}$, set $\mathcal{K} = \mathcal{K}_1 \cap \mathcal{K}_2$, and set $T_i = S \cap \mathcal{K}_i$ $(i = 1, 2)$. Let $x \in \mathcal{K}_1$. Then $x \in \mathcal{N} = \mathcal{K}_2 T$, so we may write $x = yt$ with $y \in \mathcal{K}_2$ and $t \in T$. Then $S_x = S_y$, so $(y^{-1}, x) \in \mathbf{D}$ and $y^{-1}x = t$. Thus $t \in S \cap \mathcal{K}_1\mathcal{K}_2$, and so $t \in T_2T_1$ by [He1, Theorem A]. This shows that $\mathcal{K}_1 \leq \mathcal{K}_2(T_2T_1)$. As

$$\mathcal{K}_2(T_2T_1) = (\mathcal{K}_2T_2)T_1 = \mathcal{K}_2T_1$$

by I.2.9, we obtain $\mathcal{K}_1 \leq \mathcal{K}_2T_1$. The Dedekind lemma (I.1.10) then yields

$$\mathcal{K}_1 = \mathcal{K}_1 \cap \mathcal{K}_2T_1 = (\mathcal{K}_1 \cap \mathcal{K}_2)T_1 = \mathcal{K}T_1 \leq \mathcal{K}T,$$

and so $\mathcal{K}_1T \leq \mathcal{K}T$. As $\mathcal{K}_1T = \mathcal{N}$ we conclude that $\mathcal{K}T = \mathcal{N}$, and thus $\mathcal{K} \in \mathbb{K}$. As $\mathbb{K}$ is finite, iteration of this procedure yields $O^p_{\mathcal{L}}(\mathcal{N}) \in \mathbb{K}$. The proof that $O^{p'}_{\mathcal{L}}(\mathcal{N}) \in \mathbb{K}'$ is simply the observation that $T \leq \bigcap \mathbb{K}'$. $\square$

Lemma 7.3. *Let "$*$" be either of the symbols "p" or "p', let $(\mathcal{L}, \Delta, S)$ be a proper locality, and let $(\mathcal{L}^+, \Delta^+, S)$ be an expansion of $\mathcal{L}$. Let $\mathcal{N} \trianglelefteq \mathcal{L}$, and for any partial normal subgroup $\mathcal{K} \trianglelefteq \mathcal{L}$ let $\mathcal{K}^+$ be the corresponding partial normal subgroup of $\mathcal{L}^+$ given by Theorem A2. Then $O^*_{\mathcal{L}}(\mathcal{N})^+ = O^*_{\mathcal{L}^+}(\mathcal{N}^+)$.*

Proof. Write $\mathbb{K}^+$ for the set of all $\mathcal{K}^+$ with $\mathcal{K} \in \mathbb{K}$. Then

$$(\bigcap \mathbb{K})^+ \leq \bigcap (\mathbb{K}^+)$$

as $\bigcap(\mathbb{K}^+)$ is a partial normal subgroup of $\mathcal{L}^+$ containing $\bigcap \mathbb{K}$. The reverse inclusion is given by Theorem A2, along with the observation that

$$\mathcal{L} \cap (\bigcap (\mathbb{K}^+)) = \bigcap \{\mathcal{L} \cap \mathcal{K}^+\}_{\mathcal{K} \in \mathbb{K}} = \bigcap \mathbb{K}.$$

Thus the lemma holds for "p", and the same argument applies to "p'". $\square$

Lemma 7.4. *Let $\mathcal{N}$ and $\mathcal{M}$ be partial normal subgroups of $\mathcal{L}$, with $\mathcal{N} \leq \mathcal{M}$. Let "$*$" be either of the symbols "p" or "p'. Then $O^*_{\mathcal{L}}(\mathcal{N}) \leq O^*_{\mathcal{L}}(\mathcal{M})$.*

Proof. Since $O^{p'}_{\mathcal{L}}(\mathcal{M})$ is a partial normal subgroup of $\mathcal{L}$ containg $S \cap \mathcal{N}$, we have $O^{p'}_{\mathcal{L}}(\mathcal{N}) \leq O^{p'}_{\mathcal{L}}(\mathcal{M})$ by definition. Set $\mathcal{K} = O^p_{\mathcal{L}}(\mathcal{M})$, set $\overline{\mathcal{L}} = \mathcal{L}/\mathcal{K}$, and let $\rho : \mathcal{L} \to \overline{\mathcal{L}}$ be the canonical projection. Then $(S \cap \mathcal{M})\rho = \mathcal{M}\rho \geq \mathcal{N}\rho$, and hence $\mathcal{N}\rho = (S \cap \mathcal{N})\rho$. Subgroup correspondence (I.4.7) then yields $\mathcal{N} \leq \mathcal{K}(S \cap \mathcal{N})$, and then $O^p_{\mathcal{L}}(\mathcal{N}) \leq \mathcal{K}$ by definition. $\square$

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